Extending Frobenius' Higher Characters

Monica J. Vazirani *†

Abstract. Given a finite group G with irreducible character $\chi \in \mathrm{Irr}(G)$, $r \in \mathbb{N}$ and a partition λ of r, we define higher characters $\chi_\lambda^{(r)}$ of G, following Frobenius [2]. We interpret them as a generalization of Schur functions in noncommuting variables, as a multilinear invariant map, as the sum over $\mathfrak{S}_r$ of a character of the wreath product $G \wr \mathfrak{S}_r$, and as the trace of a $e_\lambda G_r e_\lambda$-module. Using these interpretations, we are able to compute $(\chi+\psi)_\lambda^{(r)}$ in terms of $\chi_\mu^{(a)}$ and $\psi_\nu^{(b)}$ where $a+b=r$ and μ and ν are partitions of a and b, respectively.

We show distinct higher characters are orthogonal in section 3.1.

These $\chi_\lambda^{(r)}$ have the property that they are constant on $\mathfrak{S}_r \times G$ orbits of G^r, i.e. invariant under diagonal conjugation and permutation of entries of an r-tuple. By decomposing $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda) = e_\lambda G_r e_\lambda$, we find an orthogonal family of functions with this property. In the case G is abelian, we show this family forms a basis for all such functions on G^r. This doesn't happen for general G, as shown on some examples in the appendix. However, in both cases, we show that it is only necessary to consider $\lambda = (r)$ (or $\lambda = (1^r)$) in theorems 8 and 9. The reader may skip straight to section 3.2 for these results.

1 Introduction This paper is a largely expository study of Frobenius' higher characters $\chi^{(r)}$ and illuminates three different contexts in which they arise. In [2] Frobenius gives the $\chi_i^{(r)}$ in terms of the irreducible characters χ_i, as in definition 2, as well as in terms of the group determinant.

Character theory grew out of Frobenius' study of the group determinant, initiated in correspondence with Dedekind in 1896. Given a finite group G of cardinality n, choose n indeterminants $\{x_g\}_{g\in G}$ and construct the $n \times n$ matrix $X_G = [x_{gh^{-1}}]$. The group determinant is defined as $\Theta_G = det(X_G)$. Frobenius factored $\Theta_G = \prod \phi_i^{d_i}$ over $\mathbb{C}$. The ϕ_i are irreducible homogeneous polynomials of degree d_i and correspond to the irreducible complex representations ρ_i with character χ_i. One can easily recover $\chi_i(g)$ as the coefficient of $x_1^{d_i-1}$ in $\frac{\partial \phi_i}{\partial x_g}$. Further, $\phi_i = \det(\sum_{g\in G} \rho_i(g) x_g)$. But what of the other coefficients? The higher characters $\chi_i^{(r)}$ we study below arise as other coefficients occurring in the ϕ_i and their derivatives.

Recently, there has been a renewed interest in the group determinant. In 1991, Formanek and Sibley [1] proved the group determinant determines the group. In fact, Hoehnke and Johnson show we only need the $\chi_i^{(1)}, \chi_i^{(2)}$, and $\chi_i^{(3)}$ to determine the group in [4], and a result of Mansfield's gives a more elementary proof of this in [9].

Frobenius gives the higher characters $\chi_i^{(r)}$ in terms of the χ_i as in definition 2 as well as in terms of the group determinant. In this study, we use the former and a natural generalization of it to prove properties (such as orthogonality and behavior under direct sum) of the higher characters and exhibit them as occurring in other contexts of math. (Orthogonality aslo follows from considering the higher characters as coming from characters of a wreath product. [5])

1991 *Mathematics Subject Classification.* 20C05.
Key words and phrases. characters, group determinant.

We assume the reader is familiar with the basics of representation theory of finite groups and in particular with that of the symmetric group and its connection to symmetric functions [8].

Acknowledgments: I would like to thank K. W. Johnson for encouraging me to submit this, even though it was written in 1996 as a graduate student and sat on my shelf since then. Thanks to Ian Grojnowski for asking all the right questions and being infinitely generous with his time. Also, thanks to Paul Brown for teaching me GAP [11], and thanks to T.Y. Lam for his guidance and support.

2 χ_λ

2.1 Notation and some definitions Let $\mathfrak{S}_r$ denote the symmetric group on r letters. Given a permutation $\sigma \in \mathfrak{S}_r$, we can write it in cycle notation

$$\sigma = \big(m_1\ \sigma(m_1)\ \sigma^2(m_1)\cdots\sigma^{\mu_1}(m_1)\big)\big(m_2\ \sigma(m_2)\ \sigma^2(m_2)\cdots\sigma^{\mu_2}(m_2)\big)\cdots\big(m_l\ \cdots\sigma^{\mu_\ell}(m_l)\big).$$

If we use the convention that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell$, then we say σ has *shape* $\mathrm{sh}(\sigma) = (\mu_1\ \mu_2 \cdots \mu_\ell) = \mu$. Then μ is a partition of r, written $\mu \vdash r$, of length $\ell(\mu) = l$. The partition μ will also be used to denote the conjugacy class of σ which has size $\frac{r!}{z_\mu}$, where $z_\mu = |C_{\mathfrak{S}_r}(\sigma)|$. (In general, $C_G(g)$ is the centralizer of $g \in G$.) Furthermore, a partition $\lambda \vdash r$ will also be used to denote an irreducible character $\lambda \in \mathrm{Irr}(\mathfrak{S}_r)$. Let

$$e_\lambda = \frac{\lambda(1)}{r!} \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma^{-1})\sigma = \frac{\lambda(1)}{r!} \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma)\sigma$$

denote the central idempotent of $\mathbb{C}[\mathfrak{S}_r]$, which we can also view as a projection. Then $E_\lambda = \mathbb{C}[\mathfrak{S}_r]e_\lambda = e_\lambda\mathbb{C}[\mathfrak{S}_r]$ has character $\lambda(1)\lambda$. We can further decompose $e_\lambda = e_\lambda^{(1)} + e_\lambda^{(2)} + \cdots + e_\lambda^{(\lambda(1))}$ into a sum of orthogonal idempotents, obtained, for instance, by multiplying the row and column stabilizers of a standard Young tableaux of shape λ. Then $\mathbb{C}[\mathfrak{S}_r]e_\lambda^{(i)}$ is an irreducible representation of $\mathfrak{S}_r$ with character λ and its isomorphism type is independent of i. Let $L_\lambda = \mathbb{C}[\mathfrak{S}_r]e_\lambda^{(1)}$. Throughout this paper, we will use L_λ when it is most convenient to use an irreducible representation, but E_λ when we want to make use of the fact that e_λ is central.

Definition 1 *Given χ a character of G, and $\sigma \in \mathfrak{S}_r$ of shape $\mathrm{sh}(\sigma) = \mu$ in cycle notation as above, define*

$$\chi_\sigma(g_1, \ldots, g_r) = \chi(g_{m_1} g_{\sigma(m_1)} \cdots g_{\sigma^{\mu_1}(m_1)}) \cdots \chi(g_{m_l} \cdots g_{\sigma^{\mu_\ell}(m_l)}).$$

Example 1 *If $\sigma = (124)(3)$ then $\chi_\sigma(g_1, g_2, g_3, g_4) = \chi(g_1 g_2 g_4)\chi(g_3)$.*

Let sgn denote the sign character of $\mathfrak{S}_r$ that is -1 on odd permutations and 1 on even permutations.

Definition 2 *If χ is a character of G then the r-character is*

$$\chi^{(r)}(g_1, g_2, \ldots, g_r) = \sum_{\sigma \in \mathfrak{S}_r} sgn(\sigma)\chi_\sigma(g_1, g_2, \ldots, g_r).$$

A perfectly natural generalization of these higher characters is obtained by replacing $\mathrm{sgn}(\sigma)$ with any irreducible character λ of $\mathfrak{S}_r$. In fact, in [6], Johnson denotes by $\chi^{(r,+)}$ the r-character one gets using the trivial representation of $\mathfrak{S}_r$, and shows that these and the $\chi^{(r)}$ are orthogonal for fixed r, as χ ranges over $\mathrm{Irr}(G)$, the irreducible characters of G. Making this generalization, one could interpret these new r-characters as coefficients of a "group immanant". See [7].

Definition 3 *For χ an irreducible character of G and $\lambda \in \mathrm{Irr}(\mathfrak{S}_r)$ define*

$$\chi_\lambda(g_1, g_2, \ldots, g_r) = \chi_\lambda^{(r)}(g_1, g_2, \ldots, g_r) = \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma)\chi_\sigma(g_1, g_2, \ldots, g_r).$$

In section 3.1 we'll show that distinct higher characters are orthogonal.

If G has a representation V with character χ, we can interpret χ_λ as coming from the action of the wreath product of G and $\mathfrak{S}_r$ on $V^{\otimes r}$.

Let $G^r = \underbrace{G \times G \times \cdots \times G}_{r}$ and let G_r denote the *wreath product* $G \wr \mathfrak{S}_r = \mathfrak{S}_r \ltimes G^r$ where

$$(g_1, g_2, \ldots, g_r)^\sigma = \sigma^{-1}(g_1, g_2, \ldots, g_r)\sigma = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(r)}).$$

Let $\Delta(G)$ denote the copy of G that sits in G^r diagonally.

If V is an irreducible representation of G with character χ of degree n, then $V^{\otimes r}$ is an irreducible representation of G^r, with action defined by

$$(g_1, g_2, \ldots, g_r)v_1 \otimes \cdots \otimes v_r = g_1 v_1 \otimes \cdots \otimes g_r v_r,$$

and with character denoted $\chi^{\otimes r}$. The natural action of $\mathfrak{S}_r$ on $V^{\otimes r}$ via

$$\sigma\,(w_1 \otimes w_2 \otimes \cdots \otimes w_r) = w_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(2)} \otimes \cdots \otimes w_{\sigma^{-1}(r)}$$

makes $V^{\otimes r}$ into an irreducible representation of the wreath product G_r. Let us denote its character by $\widetilde{\chi}$, noticing $\mathrm{Res}^{G_r}_{G^r}\widetilde{\chi} = \chi^{\otimes r}$ and $\mathrm{Res}^{G_r}_{\Delta(G)}\widetilde{\chi} = \chi^r$.

In the notation of [5], if χ is the character of the representation V, then our $\chi_\sigma(\underline{g}) = \widetilde{\chi}(\underline{g}\sigma)$ is denoted $\chi^{(\#^n V)\widetilde{}}(\underline{g};\sigma)$ (having identified the r-tuple $\underline{g}$ with the function $\{1, \ldots, r\} \to G$ sending i to g_i). Further, $\lambda(\sigma)\chi_\sigma(\underline{g})$ is the character of $(\#^n V)\widetilde{} \otimes \lambda$ evaluated at $(\underline{g};\sigma)$. Hence the orthogonality of these extended higher characters follows from the orthogonality of the irreducible characters of the wreath product G_r.

As a way of understanding $\chi_\lambda^{(r)}$ and its connection to $\widetilde{\chi}$, let's first consider χ_λ evaluated on $\Delta(G)$. We show below that $\chi_\lambda^{(r)}(g, g, \ldots, g) = r!\chi_{S_\lambda V}(g)$, where we denote $S_\lambda V = \mathrm{Hom}_G(\mathrm{Res}^{G_r}_G \mathrm{Ind}^{G_r}_{\mathfrak{S}_r} L_\lambda, V^{\otimes r})$. Schur-Weyl duality gives us that $V^{\otimes r} \simeq \bigoplus_{\lambda \vdash r} \lambda(1) S_\lambda V$ as a $\Delta(G)$ representation. See [8] or [3]. Notice $\lambda(1)S_\lambda V \simeq e_\lambda V^{\otimes r}$. It is immediate that $S_\lambda V$ is a representation of G, and for the convenience of the reader, we compute its character.

Claim 1 *$S_\lambda V$ is a representation of G with character*

$$\chi_{S_\lambda V}(g) = \frac{1}{r!}\chi_\lambda^{(r)}(g, g, \ldots, g).$$

Proof: Recall that if $\{x_1,\ldots,x_n\}$ are the eigenvalues of g acting on V, then $\chi_{S_\lambda V}(g) = s_\lambda(x_1,\ldots,x_n)$, where s_λ denotes the Schur function. (See [8] or [3].) Notice also that g^m will have eigenvalues $\{x_1^m,\ldots,x_n^m\}$ and so

$$\chi(g^m) = x_1^m + x_2^m + \cdots + x_n^m = p_m(x_1,\ldots,x_n)$$

where p_m denotes the m^{th}-power sum symmetric function. Recall $p_\mu = p_{\mu_1}p_{\mu_2}\cdots p_{\mu_\ell}$. Then the well-known expansion of the Schur functions in terms of power sum symmetric functions gives us:

$$\begin{aligned}
\chi_{S_\lambda V}(g) &= s_\lambda(x_1,\ldots,x_n)\\
&= \sum_{\mu\vdash r}\frac{1}{z_\mu}\lambda(\mu)p_\mu(x_1,\ldots,x_n)\\
&= \frac{1}{r!}\sum_{\mu\vdash r}\frac{r!}{z_\mu}\lambda(\mu)\chi(g^{\mu_1})\chi(g^{\mu_2})\cdots\chi(g^{\mu_\ell})\\
&= \frac{1}{r!}\chi_\lambda^{(r)}(g,g,\ldots,g)
\end{aligned}$$

$\square$

This suggests we can interpret the $\chi_\lambda^{(r)}(g_1,g_2,\ldots,g_r)$ as coming from the action of the corner ring $e_\lambda\mathbb{C}[G_r]e_\lambda \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}E_\lambda, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}E_\lambda)$ on $e_\lambda V^{\otimes r} \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}E_\lambda, V^{\otimes r})$, which we will explore in section 2.3. It also suggests that $\chi_\lambda^{(r)}$ is a generalization of the Schur function s_λ.

2.2 χ_λ **as a multilinearization** Just as one can go from a bilinear form B to a quadratic form Q via $Q(x) := B(x,x)$ and then from a quadratic form to a bilinear one via $B(x,y) := \frac{1}{2}(Q(x+y)-Q(x)-Q(y))$, we can go from s_λ back up to $\chi_\lambda^{(r)}$.

First we clarify what we mean by multilinearization. Suppose we are given a homogeneous r^{th} degree polynomial $Q(\vec{x}) = Q(x_1,\ldots,x_n)$ in the (possibly noncommuting) variables $\{x_i\}_1^n$. Let $V=\mathbb{C}^n$ with standard basis $\{\vec{e_i}\}_1^n$, and let V^* have dual basis $\{\phi_i\}_1^n$. Then $\vec{x} = (x_1,\ldots,x_n) = \sum_1^n x_i\vec{e_i}$ and $\phi_i(\vec{x}) = x_i$. We'll associate to Q an element of $(V^*)^{\otimes r} \simeq (V^{\otimes r})^*$. If $Q(\vec{x}) = \sum c_{i_1 i_2\ldots i_r}x_{i_1}\cdots x_{i_r}$ let $\phi_Q = \sum c_{i_1 i_2\ldots i_r}\phi_{i_1}\otimes\phi_{i_2}\otimes\cdots\otimes\phi_{i_r}$. Notice $\phi_Q(\vec{x}\otimes\cdots\otimes\vec{x}) = Q(\vec{x})$.

We define the *multilinearization* M_Q of Q to be the homogeneous r^{th} degree polynomial in the nr noncommuting variables $\{x_{ij}\}_{\substack{1\le i\le n\\ 1\le j\le r}}$ via

$$M_Q(\vec{X}_1,\vec{X}_2,\ldots\vec{X}_r) = \frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r}\phi_Q(\vec{X}_{\sigma(1)}\otimes\vec{X}_{\sigma(2)}\otimes\cdots\otimes\vec{X}_{\sigma(r)})$$

where $\vec{X}_i = (x_{1i},x_{2i},\ldots,x_{ni})$.

One can also define M_Q more directly from Q via

$$\begin{aligned}
r!M_Q(\vec{X}_1,\ldots\vec{X}_r) &= Q(\vec{X}_1+\cdots+\vec{X}_r) - \sum_i Q(\vec{X}_1+\cdots+\widehat{\vec{X}_i}+\cdots\vec{X}_r) +\\
&\quad \sum_{i<j} Q(\vec{X}_1\cdots+\widehat{\vec{X}_i}\cdots+\widehat{\vec{X}_j}\cdots+\vec{X}_r) - \ldots\\
&\quad +(-1)^{n-1}\sum_k Q(\vec{X}_k).
\end{aligned}$$

It is tedious, but not too difficult, to verify these definitions are equivalent.

Because $\phi_Q \in (V^{\otimes r})^*$, it is easy to see M_Q is linear in each entry, i.e. r-multilinear. Also, it is clear that $Q(\vec{x}) = M_Q(\vec{x}, \vec{x}, \ldots, \vec{x})$. So we have a recipe to go from an r-multilinear form to an r-form and back.

We can multilinearize a given Q, by summing the multilinearizations of the monomials of Q. So, in the following examples, we consider just monomials.

Example

1. Let $p(x) = x^r$. Then $M_p(\vec{X}) = M_p(x_1, \ldots, x_r) = \frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r} x_{\sigma(1)}\cdots x_{\sigma(r)}$.
2. Let $q(x,y) = x^2y$. Then $M_q(x_1, y_1, x_2, y_2, x_3, y_3) =$ $\frac{1}{6}(x_1x_2y_3 + x_2x_1y_3 + x_1x_3y_2 + x_3x_1y_2 + x_2x_3y_1 + x_3x_2y_1)$.
3. Let $p(z) = z^2$. Let $r(x,y,z) = q(x,y)p(z) = x^2yz^2$. Notice $M_r(x_1, y_1, z_1, \ldots, x_5, y_5, z_5) =$ $\frac{3!2!}{5!}\sum_{\sigma\in\mathfrak{S}_5} M_q(x_{\sigma(1)}, y_{\sigma(1)}, x_{\sigma(2)}, y_{\sigma(2)}, x_{\sigma(3)}, y_{\sigma(3)})M_p(z_{\sigma(4)}, z_{\sigma(5)})$.

This process of multilinearization gives us a way to generalize symmetric functions to functions in noncommuting variables. Now let's focus on Schur functions.

Claim 2 *Let M_{s_λ} denote the multilinearization of s_λ. Then we have $s_\lambda(g) = \chi_{S_\lambda V}(g) = \chi_\lambda(g, g, \ldots, g)$ and $\chi_\lambda(g_1, g_2, \ldots, g_r) = M_{s_\lambda}(g_1, g_2, \ldots, g_r)$*

Proof: The first statement follows from claim 1. Let $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_\ell}$ denote the power sum symmetric function. If Y is an $n \times n$ matrix with eigenvalues $\{y_1, y_2, \ldots, y_n\}$, and $\lambda \vdash r$, then let us denote $p_\lambda(Y) = p_\lambda(y_1, y_2, \ldots, y_n) = \mathrm{Tr}(Y^{\lambda_1})\mathrm{Tr}(Y^{\lambda_2})\cdots\mathrm{Tr}(Y^{\lambda_l})$. Similar to the above examples, upon multilinearization we get

$$r!M_{p_\lambda}(X_1, \ldots, X_r) = \sum_{\sigma\in\mathfrak{S}_r} \mathrm{Tr}\left(X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(\lambda_1)}\right)\cdots\mathrm{Tr}\left(X_{\sigma(r-\lambda_l+1)}\cdots X_{\sigma(r)}\right).$$

Writing $\tau \in \mathfrak{S}_r$ in cycle notation $\tau = \left(\sigma(1)\sigma(2)\ldots\sigma(\lambda_1)\right)\left(\sigma(\lambda_1+1)\ldots\sigma(\lambda_1+\lambda_2)\right)\cdots\left(\sigma(r-\lambda_l+1)\ldots\sigma(r)\right)$, then the inner summand of the above is just $\mathrm{Tr}_\tau(X_1, \ldots, X_r)$.

As σ ranges over $\mathfrak{S}_r$, τ ranges over all permutations of shape λ in $\mathfrak{S}_r$ and this is a z_λ-to-1 correspondence. (In fact, all of $C_{\mathfrak{S}_r}(\tau)$ gets sent to τ.) Hence $r!M_{p_\lambda}(X_1, \ldots, X_r) = z_\lambda\sum_{\tau:sh(\tau)=\lambda}\mathrm{Tr}_\tau(X_1, \ldots, X_r)$.

So it is easy to see that the multilinearization of the Schur function $s_\lambda = \sum_{\mu\vdash r}\frac{1}{z_\mu}\lambda(\mu)p_\mu$ is just

$$\begin{aligned} M_{s_\lambda}(X_1, \ldots, X_r) &= \frac{1}{r!}\sum_{\mu\vdash r}\frac{1}{z_\mu}\lambda(\mu)\ z_\mu \sum_{\tau:sh(\tau)=\mu}\mathrm{Tr}_\tau(X_1, \ldots, X_r) \\ &= \frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r}\lambda(\sigma)\mathrm{Tr}_\sigma(X_1, \ldots, X_r). \end{aligned}$$

When $\rho : G \to GL_n(\mathbb{C})$ is a representation with character χ and we set $X_i = \rho(g_i)$, the above says the multilinearization of the Schur function evaluated at the X_i is exactly $\frac{1}{r!}\chi_\lambda^{(r)}(g_1, g_2, \ldots, g_r)$.

$\square$

Corollary 1 *If $\lambda \vdash r$ has $\ell(\lambda) > \chi(1)$, then $\chi_\lambda^{(r)}(g_1, g_2, \ldots, g_r) = 0$.*

Proof: We know that if the length of λ is greater than $n = \dim V$ then L_λ does not occur in $V^{\otimes r}$. See [8] or [3]. Hence $\chi_{S_\lambda V} = s_\lambda = 0$ and so its multilinearization is also 0.

□

We now have the machinery to compute $(\chi + \psi)_\lambda$.

Motivated by Example 3 above, we make the following definition, adopting Johnson's notation [6].

Definition 4 *For two functions $f_i : G^{r_i} \to \mathbb{C}$, let $f_1 \circ f_2 : G^{r_1+r_2} \to \mathbb{C}$ be given by*

$$f_1 \circ f_2(g_1, \ldots, g_{r_1+r_2}) = \frac{1}{r_1! r_2!} \sum_{\sigma \in \mathfrak{S}_{r_1+r_2}} f_1(g_{\sigma(1)}, \cdots g_{\sigma(r_1)}) f_2(g_{\sigma(r_1+1)}, \cdots g_{\sigma(r_1+r_2)}).$$

Remark: Notice that, up to a scalar, $f_1 \circ f_2$ multilinearizes the tensor of the functions $f_1 \cdot f_2(\underline{g}, \underline{h}) = f_1(\underline{g}) f_2(\underline{h})$.

Theorem 2 *Let $\chi, \psi \in \mathrm{Irr}(G), \lambda \vdash r$. Then*

$$(\chi + \psi)_\lambda = \sum C^\lambda_{\mu\nu} (\chi_\mu \circ \psi_\nu),$$

where $C^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.

Proof: First recall that the Littlewood-Richardson coefficients appear when we induce (or restrict) representations of $\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2}$ to $\mathfrak{S}_{r_1+r_2}$. In particular, $\mathrm{Ind}^{\mathfrak{S}_{r_1+r_2}}_{\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2}} \mu \boxtimes \nu = \sum C^\lambda_{\mu\nu} \lambda$. Using Schur Weyl duality, as in [3], we conclude that over $GL(V) \times GL(W)$

$$S_\lambda(V \oplus W) = \bigoplus C^\lambda_{\mu\nu}\, S_\mu V \otimes S_\nu W.$$

In terms of characters this says $\chi_{S_\lambda(V \oplus W)} = \sum C^\lambda_{\mu\nu}\, \chi_{S_\mu V} \cdot \chi_{S_\nu W}$. Applying claims 1 and 2 and the remark above, we see immediately that $(\chi + \psi)_\lambda = \sum C^\lambda_{\mu\nu} (\chi_\mu \circ \psi_\nu)$.

□

Corollary 3 *As previously, let $\chi^{(r)}$ denote the higher character $\chi_{(1^r)}$. Then*

$$(\chi + \psi)^{(r)} = \sum_{a+b=r} \chi^{(a)} \circ \psi^{(b)}.$$

Proof: We know the $C^{(1^r)}_{\mu\nu} = 1$ exactly when $\mu = (1^a), \nu = (1^b)$ and 0 otherwise. Or, in other words, $\wedge^r(V \oplus W) = \bigoplus_{a+b=r} \wedge^a V \otimes \wedge^b W$.

□

2.3 **χ_λ as a trace** In claim 1, we saw that $\frac{\lambda(1)}{r!} \chi^{(r)}_\lambda(g, \ldots, g)$ is the trace of the diagonal action of g on $e_\lambda V^{\otimes r}$ and therefore equals the trace of the action of $(g, g, \ldots, g) e_\lambda$ on $V^{\otimes r}$. One can then ask what is $\widetilde{\chi}((g_1, g_2, \ldots, g_r) e_\lambda)$? Although it is not a representation of G_r, $\mathrm{Hom}(\mathrm{Ind}^{G_r}_{\mathfrak{S}_r} E_\lambda, V^{\otimes r})$ is a representation of $\mathrm{Hom}(\mathrm{Ind}^{G_r}_{\mathfrak{S}_r} E_\lambda,\ \mathrm{Ind}^{G_r}_{\mathfrak{S}_r} E_\lambda)$, and we are in effect computing the trace of this action.

Claim 3 *For $\sigma \in \mathfrak{S}_r$ and $\underline{g} = (g_1, g_2, \ldots, g_r) \in G^r$, $\widetilde{\chi}(\underline{g}\sigma) = \chi_{\sigma^{-1}}(\underline{g})$.*

Proof: This is best seen on an example. Let $\sigma = (124)(3)$ and let $\underline{g} = (g_1, g_2, g_3, g_4)$. Let $\{v_i\}_1^n$ be a basis for V and write $[a_{ij}^{(k)}]$ for the matrix of g_k with respect to this basis. Then

$$\begin{aligned}
\underline{g}\sigma(v_{i_1}\otimes v_{i_2}\otimes v_{i_3}\otimes v_{i_4}) &= \underline{g}(v_{i_4}\otimes v_{i_1}\otimes v_{i_3}\otimes v_{i_2}) \\
&= \sum_{j=1}^n a_{ji_4}^{(1)} v_j \otimes \sum_{j=1}^n a_{ji_1}^{(2)} v_j \otimes \sum_{j=1}^n a_{ji_3}^{(3)} v_j \otimes \sum_{j=1}^n a_{ji_2}^{(4)} v_j \\
&= a_{i_1 i_4}^{(1)} a_{i_2 i_1}^{(2)} a_{i_3 i_3}^{(3)} a_{i_4 i_2}^{(4)} v_{i_1}\otimes v_{i_2}\otimes v_{i_3}\otimes v_{i_4} + \text{independent terms.}
\end{aligned}$$

So

$$\begin{aligned}
\widetilde{\chi}(\underline{g}\sigma) &= \sum_{i_k=1}^n a_{i_1 i_4}^{(1)} a_{i_2 i_1}^{(2)} a_{i_3 i_3}^{(3)} a_{i_4 i_2}^{(4)} \\
&= \left(\sum_{i_1=1}^n \sum_{i_4, i_2} a_{i_1 i_4}^{(1)} a_{i_4 i_2}^{(4)} a_{i_2 i_1}^{(2)}\right)\left(\sum_{i_3=1}^n a_{i_3 i_3}^{(3)}\right) \\
&= \mathrm{Tr}(a^{(1)}a^{(4)}a^{(2)})\mathrm{Tr}(a^{(3)}) = \chi(g_1 g_4 g_2)\chi(g_3) \\
&= \chi_{\sigma^{-1}}(\underline{g}).
\end{aligned}$$

The proof for general σ is similar, but for the sake of notation, we leave it to the reader.

$\square$

Theorem 4 $\widetilde{\chi}(\underline{g}e_\lambda) = \frac{\lambda(1)}{r!}\chi_\lambda^{(r)}(\underline{g})$.

Proof: From claim 3, it follows

$$\begin{aligned}
\widetilde{\chi}(\underline{g}e_\lambda) &= \frac{\lambda(1)}{r!}\sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma^{-1})\widetilde{\chi}(\underline{g}\sigma) \\
&= \frac{\lambda(1)}{r!}\sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma^{-1})\chi_{\sigma^{-1}}(\underline{g}) = \frac{\lambda(1)}{r!}\sum_{\sigma\in\mathfrak{S}_r} \lambda(\sigma)\chi_\sigma(\underline{g}) \\
&= \frac{\lambda(1)}{r!}\chi_\lambda^{(r)}(\underline{g}).
\end{aligned}$$

$\square$

In other words, $\chi_\lambda^{(r)}(\underline{g}) = \frac{r!}{\lambda(1)}\widetilde{\chi}(e_\lambda \underline{g} e_\lambda)$ (since $e_\lambda^2 = e_\lambda$), and $\widetilde{\chi}(e_\lambda \underline{g} e_\lambda)$ simply computes the trace of an element of $e_\lambda \mathbb{C}[G_r]e_\lambda \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda)$ on the module $e_\lambda V^{\otimes r} \simeq \mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} E_\lambda, V^{\otimes r})$. In section 3 we will generalize this situation by replacing $V^{\otimes r}$ with any representation of G_r.

2.4 χ_λ **as an invariant** For the moment, let's consider $G = GL(V)$. Then $S_\lambda V$ is irreducible. We can see $\chi_\lambda^{(r)}$ as a multilinear invariant map (with respect to the $\Delta(G)$-action) of $\mathrm{End}(V)^{\times r}$. The above sections 2.2 and 2.3 all translate into the language of invariants used in Procesi's work [10], which we'll outline below.

Make the identifications $(\mathrm{End}(V)^{\otimes r})^* \simeq ((V^*\otimes V)^{\otimes r})^* \simeq (V^{\otimes r})^* \otimes V^{\otimes r} \simeq \mathrm{End}(V^{\otimes r})$ and so can view a G-invariant map $\mathrm{End}(V)^{\otimes r} \to \mathbb{C}$ as an element of $\mathrm{End}(V^{\otimes r})$ which commutes with the action of $G = GL(V)$.

Again, $\mathfrak{S}_r$ acts naturally on $V^{\otimes r}$ (as does GL_r) and is centralized by the diagonal action of $G = GL(V) \simeq GL_n$. We know all operators commuting with the G-action can be realized as elements of $\mathbb{C}[\mathfrak{S}_r]$, and those commuting with both the G-action and the $\mathfrak{S}_r$-action are thus linear combinations of the e_λ.

In his theorem 1.2 of [10], Procesi shows that under the above identifications $\sigma \in \mathfrak{S}_r$ corresponds to an element he calls $\lambda_\sigma \in \mathrm{End}(V^{\otimes r})$ via

$$\lambda_\sigma : w_1 \otimes w_2 \otimes \cdots \otimes w_r \mapsto w_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(2)} \otimes \cdots \otimes w_{\sigma^{-1}(r)}$$

which in turn corresponds to an invariant $\mu_\sigma \in (\mathrm{End}(V)^{\otimes r})^*$, and that this is exactly our $\chi_{\sigma^{-1}}$. (This gives an alternate proof of claim 3.) Thus any multilinear invariant on r $n \times n$ matrices is a linear combination of the χ_σ.

Using the fact that the center of $\mathbb{C}[\mathfrak{S}_r]$ is spanned by the e_λ, we also see the invariants that commute with both the G-action and the $\mathfrak{S}_r$-action are linear combinations in the $\chi_\lambda^{(r)}$. Corollary 1 is essentially equivalent to his theorem 4.3 [10] that all "trace identities" are linear combinations of the $\chi_\lambda^{(r)}$ for which $\ell(\lambda) > n$.

3 w-functions We saw in theorem 4 that $\chi_\lambda^{(r)}(\underline{g}) = \frac{r!}{\lambda(1)}\widetilde{\chi}(\underline{g}e_\lambda)$ where $\widetilde{\chi}$ was the character of $V^{\otimes r}$ as a representation of the wreath product G_r. Let us generalize the situation to any character of G_r and examine the corresponding functions.

Definition 5 *Define the w-orbits of G^r to be the orbits of G^r under the conjugation action of $\mathfrak{S}_r \times \Delta(G)$.*

Observe these are *not* the same as the conjugacy classes in the group G_r.

Recall $\sigma^{-1}(g_1, g_2, \ldots, g_r)\sigma = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(r)})$.

Definition 6 *Let $f : G^r \to \mathbb{C}$ be any function. Then we'll call f a w-function (for lack of a better name) if*

$$f(\sigma^{-1}\underline{g}\sigma) = f(\underline{g}) = f(\underline{a}^{-1}\underline{g}\underline{a}),$$

for all $\sigma \in \mathfrak{S}_r, \underline{a} = (a, a, \ldots, a) \in \Delta(G) \subset G^r, \underline{g} = (g_1, g_2, \ldots, g_r) \in G^r$. In other words, f is constant on all w-orbits.

Definition 7 *Let χ be any character of G_r. Define $f_\lambda^\chi : G^r \to \mathbb{C}$ by*

$$f_\lambda^\chi : \underline{g} \mapsto \chi(\underline{g}e_\lambda).$$

Claim 4 *f_λ^χ is a w-function.*

Proof: First, recall χ is a class function on G_r, and e_λ and $\Delta(G)$ commute with all of $\mathbb{C}[\mathfrak{S}_r]$. Thus for $\sigma \in \mathfrak{S}_r$, $\underline{a} \in \Delta(G)$,

$$\begin{aligned} f_\lambda^\chi(\sigma^{-1}\underline{g}\sigma) &= \chi(\sigma^{-1}\underline{g}\sigma e_\lambda) = \chi(\sigma^{-1}(\underline{g}e_\lambda)\sigma) = \chi(\underline{g}e_\lambda) \\ &= f_\lambda^\chi(\underline{g}) \\ f_\lambda^\chi(\underline{a}^{-1}\underline{g}\underline{a}) &= \chi(\underline{a}^{-1}\underline{g}\underline{a}e_\lambda) = \chi(\underline{a}^{-1}(\underline{g}e_\lambda)\underline{a}) = \chi(\underline{g}e_\lambda) \\ &= f_\lambda^\chi(\underline{g}) \end{aligned}$$

□

Notice, in general, the character of any $e_\lambda\mathbb{C}[G_r]e_\lambda$-module will yield a w-function in this manner. (Warning: they are not usually class functions of G^r.) In the following sections, we will give some conditions under which the f_λ^χ are orthogonal to each other, find the span of *all* the f_λ^χ, and give a condition under which they span all the w-functions.

3.1 orthogonality

Theorem 5 *Let $\chi, \phi \in \mathrm{Irr}(G_r)$, $\lambda, \mu \in \mathrm{Irr}(\mathfrak{S}_r)$. If $\langle \lambda \cdot \chi, \mu \cdot \phi \rangle = 0$ then*

$$\sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\mu^\phi(\underline{g})} = 0.$$

Proof: Notice, because $\mathfrak{S}_r$ is a quotient of G_r we can lift λ, μ up to characters of G_r which will have G^r in their kernels. In other words, $\lambda(\underline{g}\sigma) = \lambda(\sigma), \mu(\underline{g}\sigma) = \mu(\sigma), \ \forall \underline{g} \in G^r$. Thus the inner tensor products $\lambda \cdot \chi, \mu \cdot \phi$ make sense. The orthogonality relations for characters of G_r give us that for any $\tau \in \mathfrak{S}_r \subset G_r$, $\sum_{w \in G_r} \lambda \cdot \chi(w) \overline{\mu \cdot \phi(\tau w)} = 0$. Scaling, summing over τ and reordering summations yields

$$\begin{aligned}
0 &= \tfrac{\lambda(1)}{r!} \tfrac{\mu(1)}{r!} \sum_{\tau \in \mathfrak{S}_r} \sum_{\sigma \underline{g} \in G_r} \lambda \cdot \chi(\sigma \underline{g}) \overline{\mu \cdot \phi(\tau \sigma \underline{g})} \\
&= \tfrac{\lambda(1)}{r!} \tfrac{\mu(1)}{r!} \sum_{\underline{g} \in G^r} \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma) \chi(\sigma \underline{g}) \sum_{\tau \in \mathfrak{S}_r} \overline{\mu(\tau\sigma) \phi(\tau \sigma \underline{g})} \\
&= \sum_{\underline{g} \in G^r} \chi\Big(\sum_{\sigma \in \mathfrak{S}_r} \tfrac{\lambda(1)}{r!} \lambda(\sigma) \sigma \underline{g} \Big) \overline{\phi\Big(\sum_{\tau \in \mathfrak{S}_r} \tfrac{\mu(1)}{r!} \mu(\tau\sigma) \tau \sigma \underline{g} \Big)} = \sum_{\underline{g} \in G^r} \chi(e_\lambda \underline{g}) \overline{\phi(e_\mu \underline{g})} \\
&= \sum_{\underline{g} \in G^r} \chi(\underline{g} e_\lambda) \overline{\phi(\underline{g} e_\mu)} = \sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\mu^\phi(\underline{g})}
\end{aligned}$$

□

Theorem 6 *If $\lambda \cdot \chi$ is a sum of distinct irreducible characters of G_r, then*

$$\frac{1}{|G^r|} \sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\lambda^\chi(\underline{g})} = \frac{\lambda(1)}{\chi(1)} \langle \lambda, \mathrm{Res}_{\mathfrak{S}_r}^{G_r} \chi \rangle = \frac{1}{\chi(1)} \chi(e_\lambda)$$

Proof: Now, because irreducible characters occur in $\lambda \cdot \chi$ with multiplicity 1, the relations for characters of G_r gives us $\frac{1}{|G_r|} \sum_{w \in G_r}, \lambda \cdot \chi(w) \overline{\lambda \cdot \chi(\tau w)} = \frac{1}{\lambda \cdot \chi(1)} \overline{\lambda \cdot \chi(\tau)}$. Then, using the work above, we get

$$\begin{aligned}
\frac{1}{|G^r|} \sum_{\underline{g} \in G^r} f_\lambda^\chi(\underline{g}) \overline{f_\lambda^\chi(\underline{g})} &= \frac{\lambda(1)}{\chi(1)} \frac{1}{r!} \sum_{\tau \in \mathfrak{S}_r} \lambda(\tau) \overline{\chi(\tau)} = \frac{1}{\chi(1)} \chi(e_\lambda) \\
&= \frac{\lambda(1)}{\chi(1)} \langle \lambda, \mathrm{Res}_{\mathfrak{S}_r}^{G_r} \chi \rangle.
\end{aligned}$$

□

Corollary 7 *In particular, it follows that the higher characters $\chi_\lambda^{(r)}$ are orthogonal. Further, $\frac{1}{r!} \chi_\lambda^{(r)}$ has norm $\frac{1}{n^r} s_\lambda(1, \ldots, 1)$ if it is non-zero and if $\lambda \cdot \widetilde{\chi}$ consists of distinct characters.*

Proof As before, if $\chi_i \in \mathrm{Irr}(G)$ is the character of a representation V_i, then let $\widetilde{\chi_i} \in \mathrm{Irr}(G_r)$ be the irreducible character of the associated action of G_r on $V_i^{\otimes r}$. Notice that

$$\langle \mathrm{Res}_{G^r}^{G_r} \lambda \cdot \widetilde{\chi_i}, \mathrm{Res}_{G^r}^{G_r} \mu \cdot \widetilde{\chi_j} \rangle_{G^r} = \lambda(1)\mu(1) \langle \chi_i^{\otimes r}, \chi_j^{\otimes r} \rangle_{G^r} = 0$$

if $i \neq j$. Thus we must also have $\langle \lambda \cdot \widetilde{\chi}_i, \mu \cdot \widetilde{\chi_j} \rangle_{G_r} = 0$, and the result follows from theorem 5. Observe that we know the action of $\sigma \in \mathfrak{S}_r$ on $V^{\otimes r}$, so that $\widetilde{\chi}(\sigma) = n^{\ell(\mathrm{sh}(\sigma))}$ where $n = \chi(1)$. Hence, if $\lambda \cdot \widetilde{\chi}$ is a sum of distinct irreducible characters, then theorems 4 and 6 give that the norm of $\frac{1}{r!}\chi_\lambda^{(r)}$ is $\frac{1}{\widetilde{\chi}(1)}\frac{1}{r!}\sum_{\tau \in \mathfrak{S}_r} \lambda(\tau) n^{\ell(\mathrm{sh}(\tau))} = \frac{1}{n^r} s_\lambda(1, 1, \ldots, 1)$.

□

Notice, when $\chi \in \mathrm{Irr}(G_r)$ has $\mathrm{Res}^{G_r}_{G^r}\chi$ irreducible, i.e. the restriction is of the form $\psi_1^{\mu_1} \otimes \cdots \otimes \psi_\ell^{\mu_\ell}$ for distinct $\psi_i \in \mathrm{Irr}(G), \mu \vdash r$, then the f_λ^χ are symmetrized products of the ${\psi_i}_\lambda^{(\mu_i)}$ (as with the multilinearizations in the examples of section 2.2), and so also include the functions denoted by $\chi_i \circ \chi_j$ in Johnson's work [6]. The above proof extends to show that this larger class of f_λ^χ are orthogonal.

3.2 counting w-orbits and w-functions

Claim 5 *The dimension of the space of all w-functions on a finite group G is*

$$\frac{1}{|G|}\sum_{g \in G}\sum_{\mu \vdash r}\frac{1}{z_\mu}|C_G(g^{\mu_1})| \cdots |C_G(g^{\mu_\ell})|.$$

Proof: Counting the dimension of the space of w-functions is equivalent to counting the number of orbits of the conjugation action of $\mathfrak{S}_r \times \Delta(G)$ on G^r. We count them as follows. G acts on G by conjugation with character $\psi(g) = |C_G(g)|$. G acts on G^r under the diagonal action with character $\psi^{\otimes r}$. Asking how many $\mathfrak{S}_r$ orbits there are is the same as asking how many times the trivial character occurs in $Sym^r(\mathbb{C}[G]) = S_{(r)}(\mathbb{C}[G]) = S_1(\mathbb{C}[G])$. We know the character of G on $S_1(\mathbb{C}[G])$ is just $\psi_1^{(r)}|_{\Delta(G)}$, recalling $\lambda = (r)$ denotes the trivial representation of $\mathfrak{S}_r$, which by abuse of notation we also sometimes refer to as 1. Hence the number of w-orbits is

$$\begin{aligned}\langle 1, \psi_1^{(r)} \rangle_{\Delta(G)} &= \frac{1}{|G|}\sum_{g \in G}\sum_{\mu \vdash r}\frac{1}{z_\mu}\psi(g^{\mu_1})\psi(g^{\mu_2})\cdots\psi(g^{\mu_\ell}) \\ &= \frac{1}{|G|}\sum_{g \in G}\sum_{\mu \vdash r}\frac{1}{z_\mu}|C_G(g^{\mu_1})| \cdots |C_G(g^{\mu_\ell})|.\end{aligned}$$

□

Theorem 8 *Let $B = \{f_1^\chi : \chi(e_1) \neq 0\}$. Then B is an orthogonal basis for span $\{f_\lambda^\chi : \lambda \in \mathrm{Irr}(\mathfrak{S}_r), \chi \in \mathrm{Irr}(G_r)\}$. In particular, the higher characters are all linear combinations of the f_1^χ.*

Proof: By abuse of notation, we write 1 for the trivial representation $\lambda = (r) \in \mathrm{Irr}(\mathfrak{S}_r)$. Theorem 5 gives us that the f_1^χ are orthogonal. Take any $f_\lambda^\chi : \underline{g} \mapsto \chi(\underline{g}\sigma)$. Recall (as in the proof of theorem 5) we can lift λ up to a character of G_r, also denoted λ, which will have G^r in its kernel.

Write $\lambda \cdot \chi = \sum_{\chi_i \in \mathrm{Irr}(G_r)} m_i \chi_i$. Then

$$\begin{aligned}\chi(\underline{g}e_\lambda) &= \chi(\underline{g}\sum_{\sigma \in \mathfrak{S}_r}\lambda(\sigma)\sigma) = \sum_{\sigma \in \mathfrak{S}_r}\lambda(\sigma)\chi(\underline{g}\sigma) = \sum_{\sigma \in \mathfrak{S}_r}\lambda(\underline{g}\sigma)\chi(\underline{g}\sigma) \\ &= \sum_{\sigma \in \mathfrak{S}_r}\lambda \cdot \chi(\underline{g}\sigma) = \lambda \cdot \chi(\underline{g}\sum_{\sigma \in \mathfrak{S}_r}\sigma) = \lambda \cdot \chi(\underline{g}e_1) \\ &= \sum_{\chi_i \in \mathrm{Irr}(G_r)} m_i \chi_i(\underline{g}e_1).\end{aligned}$$

Hence we see $f_\lambda^\chi = \sum_{\chi_i \in \mathrm{Irr}(G_r)} m_i f_1^\chi$. By the remark below, $f_1^\chi = 0$ if $f_1^\chi \notin B$, thus $f_\lambda^\chi \in span(B)$.

□

Remark: Notice, the set $\{\chi \in \mathrm{Irr}(G_r) : \chi(e_\lambda) \neq 0\}$ are exactly the support of $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} L_\lambda$, since

$$\chi(e_\lambda) = \frac{\lambda(1)}{r!} \sum_{\sigma \in \mathfrak{S}_r} \lambda(\sigma)\chi(\sigma) = \lambda(1)\langle \mathrm{Res}_{\mathfrak{S}_r}^{G_r} \chi, \lambda \rangle_{\mathfrak{S}_r} = \lambda(1)\langle \chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} \lambda \rangle.$$

Let U be the representation of G_r with character χ. Then U and $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1, U)$ are isomorphic as $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1)$-modules and f_1^χ is the trace of this action. When $\chi(e_1) = 0$ then $\mathrm{Hom}_{G_r}(\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1, U) = 0$, and so its trace f_1^χ is also 0.
Remark: We could also replace the trivial character $\lambda = (r)$ by the sign character $\lambda = (1^r)$ in theorem 8 and theorem 9 (to follow).

3.3 abelian G By comparing the number of w-orbits to the size of the support of $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} 1$, we see that in general not all w-functions occur as f_λ^χ. For example, when $r = 2, G = \mathfrak{S}_3$ there are 8 w-orbits but $|B| = 7$. When $r = 3, G = \mathfrak{S}_3$ there are 17 w-orbits but $|B| = 13$. See the appendix. However, when G is abelian, we do indeed capture all the w-functions. To prove this, we require a simple case of the following lemma.

Lemma 1 *Let $\lambda \in \mathrm{Irr}(\mathfrak{S}_r), \underline{h} \in G^r$, and $\tau \in \mathfrak{S}_r$ be of shape $sh(\tau) = \mu$. Define $\underline{h}_\tau = (h_{m_1} h_{\tau(m_1)} \cdots h_{\tau^{\mu_1}(m_1)}, \ldots, h_{m_l} h_{\tau(m_l)} \cdots h_{\tau^{\mu_\ell}(m_l)})$. Then*

$$\mathrm{Ind}_{\mathfrak{S}_r}^{G_r} \lambda(\tau \underline{h}) = \begin{cases} \lambda(\tau)|G|^{\ell(\mu)} & \text{if } \underline{h}_\tau = (1^{\ell(\mu)}) \\ 0 & \text{otherwise} \end{cases}$$

Proof:

$$\begin{aligned} \mathrm{Ind}_{\mathfrak{S}_r}^{G_r} \lambda(\tau \underline{h}) &= \frac{1}{r!} \sum_{\substack{\sigma \underline{g} \in G_r : \\ (\sigma \underline{g}) \tau \underline{h} (\sigma \underline{g})^{-1} \in \mathfrak{S}_r}} \lambda((\sigma \underline{g}) \tau \underline{h} (\sigma \underline{g})^{-1}) \\ &= \frac{1}{r!} \sum_{\substack{\sigma \underline{g} \in G_r : \\ \sigma\tau\sigma^{-1}[\underline{g}^\tau \underline{h} \underline{g}^{-1}]^{\sigma^{-1}} \in \mathfrak{S}_r}} \lambda(\sigma\tau\sigma^{-1}[\underline{g}^\tau \underline{h} \underline{g}^{-1}]^{\sigma^{-1}}) \\ &= \sum_{\substack{\underline{g} \in G^r : \\ \underline{g}^\tau \underline{h} \underline{g}^{-1} = \underline{1}}} \lambda(\tau) \end{aligned}$$

So, we'll be done if we can show that $\#\{\underline{g} \in G^r : \underline{g}^\tau \underline{h} \underline{g}^{-1} = \underline{1}\} = |G|^{\ell(\mu)}$ when $\underline{h}_\tau = (1, 1, \ldots, 1) = (1^{\ell(\mu)})$, and 0 otherwise. For instance, we have $|G|$-many choices for g_{m_1}. The equation $\underline{g}^\tau \underline{h} \underline{g}^{-1} = \underline{1}$ then specifies for us the values of $g_{\tau^q(m_1)}$, and also imposes the requirement that the product $h_{m_1} h_{\tau(m_1)} \cdots h_{\tau^{\mu_1}(m_1)} = 1$. The same argument holds for $m_2, \ldots, m_{\ell(\mu)}$. Our requirement is met for all the m_j precisely when $\underline{h}_\tau = (1^{\ell(\mu)})$, and in this case, we have $|G|^{\ell(\mu)}$ choices for $\underline{g}$.

□

Theorem 9 *If G is abelian, then $B = \{f_1^\chi : \chi(e_1) \neq 0\}$ is an orthogonal basis for the space of all w-function on G^r.*

Proof: We simply do a dimension count. We saw in claim 5 that the number of w-orbits for an abelian group is $\sum_{\lambda\vdash r}\frac{1}{z_\lambda}|G|^{\ell(\lambda)}$. We also computed in lemma 1 that $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1(\tau) = |G|^{\ell(sh(\tau))}$. So

$$\begin{aligned}
\langle \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle &= \langle \mathrm{Res}_{\mathfrak{S}_r}^{G_r}\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1, 1\rangle_{\mathfrak{S}_r} = \frac{1}{r!}\sum_{\tau\in\mathfrak{S}_r}\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1(\tau)\ \overline{1(\tau)} \\
&= \frac{1}{r!}\sum_{\tau\in\mathfrak{S}_r}|G|^{\ell(sh(\tau))} \\
&= \sum_{\lambda\vdash r}\frac{1}{z_\lambda}|G|^{\ell(\lambda)}.
\end{aligned}$$

Next we'll show that if $\chi(e_1)\neq 0$ then $\langle\chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle = 1$. In other words, it occurs with multiplicity one. Take any $\chi\in\mathrm{Irr}(G_r)$. Since G is abelian, χ can be realized as $\mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi$ where $\phi\in\mathrm{Irr}(G^r)$, $\rho\in\mathrm{Irr}(H)$, and $H\subset\mathfrak{S}_r$ is the stabilizer of ϕ with action given by $\sigma\cdot\phi(\underline{g}) = \phi(\sigma^{-1}\underline{g}\sigma)$. Both ϕ and ρ extend to characters of $H\ltimes G^r$, because H is the stabilizer of ϕ and because H is a quotient of $H\ltimes G^r$. Using Mackey's criterion, we see $\mathrm{Res}_{\mathfrak{S}_r}^{G_r}\mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi = \mathrm{Ind}_H^{\mathfrak{S}_r}\rho$. Hence

$$\begin{aligned}
\langle\chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle &= \langle \mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle = \langle \mathrm{Res}_{\mathfrak{S}_r}^{G_r}\mathrm{Ind}_{H\ltimes G^r}^{G_r}\rho\cdot\phi, 1\rangle \\
&= \langle \mathrm{Ind}_H^{\mathfrak{S}_r}\rho, 1\rangle_{\mathfrak{S}_r} = \langle\rho, \mathrm{Res}_H^{\mathfrak{S}_r}1\rangle_H = \langle\rho, 1\rangle_H \\
&= 1 \text{ or } 0
\end{aligned}$$

We conclude $|B| = \#\{\chi\in\mathrm{Irr}(G_r) : \chi(e_1)\neq 0\} = \#$ of χ in the support of $\mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1$ $= \langle \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle = \#$ of w-orbits.

□

Remark: Since characters of wreath products of the form G_r are well understood [8] and the combinatorics of the character theory of $\mathfrak{S}_n$ is well-understood, one could combine the two to get combinatorial formulas for the higher characters of $\mathfrak{S}_n$. We have computed some small examples, and as yet saw no structure in the data. (These computations were done in 1996-97, so perhaps more recent literature of which the author is unaware addresses this structure.)

4 Appendix In this appendix, we list the non-zero f_1^χ for $\chi\in\mathrm{Irr}(G)$ where $G=\mathfrak{S}_3$, $r=2, r=3$. The computations were done using the package GAP [11].

The first column lists w-orbit representatives, where $\tau=(23)$, $\sigma=(123)$. The second column is the size of each w-orbit. The very last row of the table gives the norm of f_1^χ. Notice the first row is just $\chi(e_1) = \langle\chi, \mathrm{Ind}_{\mathfrak{S}_r}^{G_r}1\rangle$, and so one can read off the degree $\chi(1)$ by dividing the first row by the last.

The higher characters $\psi_\lambda^{(r)}$ for $\psi\in\mathrm{Irr}(\mathfrak{S}_3), \lambda\vdash r$ are contained in the table as well. Write $\mathrm{Irr}(\mathfrak{S}_3) = \{1, sgn, \mu\}$.

Then for $r=2$ (Table I.), the first column is $\frac{1}{2!}1_{(2)} = 1\otimes 1$, the second column is $\frac{1}{2!}sgn_{(2)} = sgn\otimes sgn$, the fourth column is $\frac{1}{2!}\mu_{(2)}$, and the fifth column is $\frac{1}{2!}sgn_{(11)}$. Notice the sum of the fourth and fifth columns is $\mu\otimes\mu\in\mathrm{Irr}(\mathfrak{S}_3^2)$.

For $r=3$ (Table II.), the first column is $\frac{1}{3!}1_{(3)} = 1\otimes 1\otimes 1$, the second column is $\frac{1}{3!}sgn_{(3)} = sgn\otimes sgn\otimes sgn$, the seventh column is $\frac{1}{3!}\mu_{(3)}$, and the thirteenth column is $\frac{1}{2}\frac{1}{3!}sgn_{(21)}$. Notice the second column plus twice the thirteenth column is $\mu\otimes\mu\otimes\mu\in\mathrm{Irr}(\mathfrak{S}_3^3)$.

I.

The f_1^χ for $G = \mathfrak{S}_3, r = 2$

$(1,1)$	1	1	1	1	3	1	2	2
$(1,\sigma\tau)$	6	1	-1	0	0	0	1	-1
$(1,\sigma)$	4	1	1	1	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
(τ,τ)	3	1	1	-1	1	-1	0	0
$(\tau,\sigma\tau)$	6	1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
(τ,σ)	12	1	-1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
(σ,σ)	2	1	1	1	0	1	-1	-1
(σ,σ^{-1})	2	1	1	1	$\frac{3}{2}$	$-\frac{1}{2}$	-1	-1
norm		1	1	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

II.

The f_1^χ for $G = \mathfrak{S}_3, r = 3$

$(1,1,1)$	1	1	1	1	1	2	2	4	2	1	3	1	3	2
$(1,1,\tau)$	9	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$	$-\frac{4}{3}$	0	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	-1	0
$(1,1,\sigma)$	6	1	1	1	1	1	1	-2	1	0	0	0	0	-1
$(1,\tau,\tau)$	9	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$
$(1,\tau,\sigma\tau)$	18	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
$(1,\tau,\sigma)$	36	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	$-\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	0
$(1,\sigma,\sigma)$	6	1	1	1	1	0	0	0	0	0	-1	0	-1	1
$(1,\sigma,\sigma^{-1})$	6	1	1	1	1	0	0	2	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
(τ,τ,τ)	3	1	-1	-1	1	0	0	0	0	-1	1	1	-1	0
$(\tau,\tau,\sigma\tau)$	18	1	-1	-1	1	0	0	0	0	0	0	0	0	0
(τ,τ,σ)	18	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$(\tau,\sigma\tau,\sigma)$	36	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$(\tau,\sigma\tau,\tau\sigma)$	6	1	-1	-1	1	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
(τ,σ,σ)	18	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	0
$(\tau,\sigma,\sigma^{-1})$	18	1	-1	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	0	0	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	0
(σ,σ,σ)	2	1	1	1	1	-1	-1	1	-1	1	0	1	0	-1
$(\sigma,\sigma,\sigma^{-1})$	6	1	1	1	1	-1	-1	-1	-1	0	1	0	1	0
norm		1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{8}$

References

[1] E. Formanek, D. Sibley, "The group determinant determines the group," *Proc. Amer. Math. Soc.* **112** (1991), 649-656.

[2] F. G. Frobenius, "Über die Primfactoren der Gruppendeterminante," *S'Ber. Akad. Wiss. Berlin* (1896) 1343-1382. (*Ges. Abh.* III, Springer-Verlag, Berlin, 1968, pp. 38-77.)

[3] W. Fulton and J. Harris, *Representation Theory: A First Course,* Graduate Texts in Mathematics **129**, Springer-Verlag, New York, 1991. (Lecture 6)

[4] H.J. Hoehnke, K.W. Johnson, "The 1- 2- 3- characters are sufficient to determine a group," *Bull. Amer. Math. Soc.* **27** (1992), 243-245.

[5] G. James and A. Kerber, *The Representation Theory of the Symmetric Group,* (Encyclopedia of Mathematics and Its Applications, Vol 16), Addison-Wesley, Reading, 1981.

[6] K. W. Johnson, "On the group determinant," *Math. Proc. Camb. Phil. Soc.* **109** (1991), 299-311.

[7] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups,* 2nd ed., Clarendon Press, Oxford, 1940.

[8] I. G. MacDonald, *Symmetric Functions and Hall Polynomials* 2nd ed., Clarendon Press, Oxford, 1995.

[9] R. Mansfield, "A group determinant determines its group," *Proc. Amer. Math. Soc.* **116** (1992), 939-941.

[10] C. Procesi, "The invariant theory of $n \times n$ matrices," *Adv. in Math.* **19**, (1976), 306-381.

[11] M. Schönert et al. *GAP – Groups, Algorithms, and Programming.* Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth edition, 1995.