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An observation on highest weight crystals

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Abstract

In this paper, we observe that a certain local property on highest weight crystal graphs forces a more global property. In type A, this statement says that if a node has a single parent and single grandparent, then there is a unique walk from the highest weight node to it. This crystal observation was motivated by certain representation-theoretic behavior of the affine Hecke algebra of type A. In other classical types, there is a similar statement. This walk is obtained from the associated level 1 perfect crystal, $B^{1,1}$. (It is unique unless the Dynkin diagram contains that of D_4 as a subdiagram.)

Keywords: Crystal graphs

1. Introduction

As shown in the paper of Stembridge [1], crystal graphs can be characterized by their local behavior. In this paper, we observe that a certain local property on crystals forces a more global property. In type A, this statement says that if a node has a single parent and single grandparent, then there is a unique walk from the highest weight node to it. In other classical types, there is a similar (but necessarily more technical) statement. This walk is obtained from the associated level 1 perfect crystal, $B^{1,1}$. (It is unique unless the Dynkin diagram contains that of D_4 as a subdiagram.)

This crystal observation was motivated by certain representation-theoretic behavior of the affine Hecke algebra of type A, which is known to be captured by highest weight crystals of type

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$A^{(1)}$ by the results in [2]. As discussed below, the proofs in either setting are straightforward, and so Grojnowski's theorem linking the two phenomena is not needed. However, the result is presented here for crystals as one can say something in all types (Grojnowski's theorem is only in type A), and because the statement seems more surprising in the language of crystals than it does for affine Hecke algebra modules.

Here, we describe some of the representation-theoretic phenomena that motivated the main theorem (Theorem 3.6), and expand on this further in Section 5.

One may ask which irreducible S_n-modules are still irreducible upon restriction to S_{n-2}. In characteristic 0, the Branching Rule makes this quite easy to solve combinatorially. For the restriction of an irreducible module from S_n to S_{n-1} to be irreducible means its associated partition can have at most one removable box, and hence must be a rectangle. But for that rectangle to share the same property, the original shape must have been a single row or column, hence our original representation is the trivial or sign module. In particular, it is one-dimensional; thus its restriction to any S_{n-k} is also irreducible. In Section 5 we give another proof that does not rely on the Branching Rule (nor indeed on a combinatorial parameterization of the irreducibles) and which works in any characteristic. In fact, the proof is actually given for the more general case of representations of the affine Hecke algebra. The answer is still that our original representation is one-dimensional, and so a "trivial" or "sign" module.

When this same statement is translated into the language of crystals, it becomes case (1) (type A) of Theorem 3.7. Under this translation, nodes in the crystal correspond to simple modules and "in" arrows correspond to not just restriction, but the simple submodules of restriction. In the crystal setting, if a node in a highest weight crystal has a single parent and single grandparent, then there is a unique walk from the highest weight node to it. Equivalently, the weight labeling the collection of "in" arrows to a node on this walk has level 1. It is then natural to ask what behavior such nodes exhibit in crystals of other type. This is described in full in the rest of Theorem 3.7.

2. Crystals

We begin by reviewing some of the definitions and notation for crystal graphs, but assume the reader is familiar with crystals and with root systems. For a more comprehensive and complete discussion, see [3].

In the following, we fix a root system of finite or affine type. I indexes the simple roots (and the nodes of the corresponding Dynkin diagram); P is the weight lattice; P^* is the coroot lattice with canonical pairing $\langle\ ,\ \rangle$. The simple roots are $\alpha_i \in P$, and simple coroots are $h_i \in P^*$. The fundamental weights are denoted Λ_i and satisfy $\langle h_i, \Lambda_j\rangle = \delta_{ij}$. The matrix $[a_{ij}]$ where $a_{ij} = \langle h_i, \alpha_j\rangle$ is the corresponding Cartan matrix.

A *crystal* is a set of nodes B, endowed with the following maps

$$\begin{aligned}
&\mathrm{wt} : B \to P,\\
&\varepsilon_i : B \to \mathbb{Z} \cup \{-\infty\},\\
&\varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\},\\
&\tilde{e}_i : B \to B \sqcup \{\mathbf{0}\},\\
&\tilde{f}_i : B \to B \sqcup \{\mathbf{0}\}.
\end{aligned}$$

The maps satisfy the following axioms:

$$\varphi_i(\mathbf{b}) = \varepsilon_i(\mathbf{b}) + \langle h_i, \mathrm{wt}(\mathbf{b})\rangle \quad \forall i \in I,\ \mathbf{b} \in B.$$
If $\tilde{e}_i\mathbf{b} \neq \mathbf{0}$, then $\varepsilon_i(\tilde{e}_i\mathbf{b}) = \varepsilon_i(\mathbf{b}) - 1,\ \varphi_i(\tilde{e}_i\mathbf{b}) = \varphi_i(\mathbf{b}) + 1,\ \mathrm{wt}(\tilde{e}_i\mathbf{b}) = \mathrm{wt}(\mathbf{b}) + \alpha_i$.
If $\tilde{f}_i\mathbf{b} \neq \mathbf{0}$, then $\varepsilon_i(\tilde{f}_i\mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1,\ \varphi_i(\tilde{f}_i\mathbf{b}) = \varphi_i(\mathbf{b}) - 1,\ \mathrm{wt}(\tilde{f}_i\mathbf{b}) = \mathrm{wt}(\mathbf{b}) - \alpha_i$.
For $\mathbf{a}, \mathbf{b} \in B$, $\mathbf{a} = \tilde{f}_i\mathbf{b}$ if and only if $\mathbf{b} = \tilde{e}_i\mathbf{a}$.
If $\varphi_i(\mathbf{b}) = -\infty$, then $\tilde{e}_i\mathbf{b} = \tilde{f}_i\mathbf{b} = \mathbf{0}$.

Given the crystal data, we can draw the associated *crystal graph*. It is a directed graph with nodes B, and I-colored arrows given by

$$\mathbf{b} \xrightarrow{i} \mathbf{a}$$

when $\mathbf{a} = \tilde{f}_i\mathbf{b}$, or equivalently when $\mathbf{b} = \tilde{e}_i\mathbf{a}$.

In all of the following, we will make the extra assumption that our crystal B is a highest weight crystal. Consequently, we can read the data of

$$\varepsilon_i(\mathbf{b}) = \max\{n \geqslant 0 \mid \tilde{e}_i^n\mathbf{b} \neq \mathbf{0}\},$$
$$\varphi_i(\mathbf{b}) = \max\{n \geqslant 0 \mid \tilde{f}_i^n\mathbf{b} \neq \mathbf{0}\},$$

off of the crystal graph, encoded in the following picture

$$\underbrace{\bullet \xrightarrow{i} \bullet \xrightarrow{i} \cdots \bullet \xrightarrow{i}}_{\varepsilon_i(\mathbf{b})} \mathbf{b} \underbrace{\xrightarrow{i} \bullet \xrightarrow{i} \cdots \xrightarrow{i}}_{\varphi_i(\mathbf{b})} \bullet.$$

We will also use the notation $\varepsilon(\mathbf{b}) = \sum_{i\in I}\varepsilon_i(\mathbf{b})\Lambda_i$. Thus $\varepsilon(\mathbf{b})$ describes the "in"-arrows leading to the node $\mathbf{b}$.

Below, we will be interested in describing certain cases where $\varepsilon(\mathbf{b}) = \Lambda_i$ and $\varepsilon(\tilde{e}_i\mathbf{b}) = \Lambda_j$. (However we will not put any restrictions on "out"-arrows.)

2.1. Extra terminology

We introduce some terminology below.

Let us say that a node $\mathbf{a}$ is *singular* if

$$\sum_{i\in I} m_i \leqslant 1, \quad \text{where } \varepsilon(\mathbf{a}) = \sum_{i\in I} m_i\Lambda_i. \tag{2.1}$$

Notice Eq. (2.1) implies there is at most one $i \in I$ such that $\tilde{e}_i\mathbf{a} \neq \mathbf{0}$. In particular, highest weight nodes satisfy (2.1). In the crystal graph, we picture singular nodes as having a single "in"-arrow leading to it (and any arrow preceding that one carries a different color), but there is no restriction on its "out"-arrows.

If $\tilde{e}_i(\mathbf{a}) = \mathbf{b}$ for some i, we shall say $\mathbf{b}$ is a *parent* of $\mathbf{a}$. We will define *ancestor* inductively by saying parents are ancestors and parents of ancestors are also ancestors.

Note that being singular is a weaker condition that requiring that $\varepsilon(\mathbf{a})$ has level $\leqslant 1$ (but stronger than having level $\leqslant 2$), where in the affine case, level is given by $\langle \mathfrak{c}, \varepsilon(\mathbf{a})\rangle$, where $\mathfrak{c}$ is the central element. However, it will occasionally be useful to characterize nodes **a** by the level of $\varepsilon(\mathbf{a})$. For the algebras we are considering here, always $\langle \mathfrak{c}, \Lambda_i \rangle \in \{1, 2\}$. We can also talk about the level of a fundamental weight in the non-affine cases, by embedding the finite algebra in its affinization in the natural way.

3. Kashiwara's theorem for highest weight crystals

In all the following theorems, we fix a root system and assume B is a fixed highest weight crystal of that type.

The crystal graph B comes from an integrable highest weight module V of the associated Lie algebra or quantum enveloping algebra. We appeal to theorems of Kashiwara that ensure the existence of a global basis $\{G(\mathbf{b}) \mid \mathbf{b} \in B\}$ of V. In the following e_i will denote a Chevalley generator, and $e_i^{(m)}$ its divided power.

We first give a remark (in Section 5) of [4] as the following useful lemma. One should compare it to the statement $\mathrm{wt}(\tilde{e}_i \mathbf{b}) = \mathrm{wt}(\mathbf{b}) + \alpha_i$.

Lemma 3.1. *When $\tilde{e}_i \mathbf{b} \neq \mathbf{0}$,*

$$\varepsilon(\tilde{e}_i \mathbf{b}) = \varepsilon(\mathbf{b}) + \sum_{j \in I} m_j \Lambda_j, \quad \text{where } m_i = -1,\ 0 \leqslant m_j \leqslant -a_{ij}. \tag{3.1}$$

In general, we have no control over the value m_j takes in the range $0 \leqslant m_j \leqslant -a_{ij}$ for $j \neq i$. Below, we will be interested in describing certain cases where we can force a single $m_j = 1$ and the rest zero. In other words, we want that $\varepsilon(\mathbf{b}) = \Lambda_i$ and $\varepsilon(\tilde{e}_i \mathbf{b}) = \Lambda_j$.

We list some immediate corollaries to this lemma.

Corollary 3.2. *Let $\mathbf{b} \in B$ and suppose $a_{ij} = 0$. Then $\tilde{e}_j \mathbf{b} = \mathbf{0} \Rightarrow \tilde{e}_j \tilde{e}_i \mathbf{b} = \mathbf{0}$.*

Corollary 3.3. *Let $\mathbf{a}, \mathbf{b} \in B$ both be singular nodes, and suppose $\mathbf{b}$ is a parent of $\mathbf{a}$, with $\mathbf{b} = \tilde{e}_i \mathbf{a}$. Then $\tilde{e}_j \mathbf{b} \neq \mathbf{0} \Rightarrow a_{ij} < 0$.*

Theorem 3.4. *(See [4].) Let $\mathbf{b} \in B$ and suppose $\tilde{e}_i^m \mathbf{b} \neq \mathbf{0}$ but $\tilde{e}_i^{m+1} \mathbf{b} = \mathbf{0}$. Then*

$$e_i^{(m)} G(\mathbf{b}) = G\big(\tilde{e}_i^m \mathbf{b}\big) \quad \text{and} \quad e_i^{(m+1)} G(\mathbf{b}) = 0.$$

As a corollary to this theorem, employing the Serre relations, we can deduce several properties of singular nodes. We review the Serre relations below.

Fix $i, j \in I, i \neq j$. Let $\ell = 1 - \langle h_i, \alpha_j \rangle = 1 - a_{ij}$. Then

$$\sum_{k=0}^{\ell} (-1)^k e_i^{(k)} e_j e_i^{(\ell-k)} = 0. \tag{3.2}$$

Corollary 3.5.

(1) *Suppose $a_{ij} = 0$. Then $\tilde{e}_j \mathbf{b} = \mathbf{0}$, $\tilde{e}_i^2 \mathbf{b} = \mathbf{0} \Rightarrow \tilde{e}_j(\tilde{e}_i \mathbf{b}) = \mathbf{0}$.*
(2) *Suppose $a_{ij} = -1$. Then $\tilde{e}_j \mathbf{b} = \mathbf{0}$, $\tilde{e}_i^2 \mathbf{b} = \mathbf{0} \Rightarrow \tilde{e}_i(\tilde{e}_j \tilde{e}_i \mathbf{b}) = \mathbf{0}$.*

(3) *Suppose* $a_{ij} = -2$. *Then* $\tilde{e}_j \mathbf{b} = \mathbf{0}, \tilde{e}_i^2 \mathbf{b} = \mathbf{0}$, $\tilde{e}_j^2 \tilde{e}_i \mathbf{b} = \mathbf{0} \Rightarrow \tilde{e}_i(\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b}) = \mathbf{0}$. *Also* $\tilde{e}_i b = \mathbf{0}$, $\tilde{e}_j^2 \mathbf{b} = \mathbf{0} \Rightarrow \tilde{e}_i^3 \tilde{e}_j \mathbf{b} = \mathbf{0}$.
If in addition $a_{ji} = -1$, *then* $\tilde{e}_j b = \mathbf{0}$, $\tilde{e}_i^2 \mathbf{b} = \mathbf{0} \Rightarrow \tilde{e}_j^2 \tilde{e}_i \mathbf{b} = \mathbf{0}$ *and* $\tilde{e}_j(\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b}) = \mathbf{0}$.

Proof. (1) This follows directly from Corollary 3.2, which is a stronger statement. (We note one may also prove this using Theorem 3.4 in a manner similar to the subsequent cases.)

(2) From the Serre relations for $a_{ij} = -1$, we know that $(e_i^{(2)} e_j - e_i e_j e_i + e_j e_i^{(2)})(G(\mathbf{b})) = 0$. Applying Theorem 3.4, $\tilde{e}_j \mathbf{b} = \mathbf{0} \Rightarrow e_j G(\mathbf{b}) = 0$ and $\tilde{e}_i^2 \mathbf{b} = \mathbf{0}$ implies both that $e_i^{(2)} G(\mathbf{b}) = 0$ and $e_i G(\mathbf{b}) = G(\tilde{e}_i \mathbf{b})$. Hence we get $0 = e_i e_j e_i G(\mathbf{b}) = e_i e_j G(\tilde{e}_i \mathbf{b})$.

Kashiwara's equation (5.3.8) in [4] gives $e_i G(\mathbf{b})$ as a linear combination of $G(\tilde{e}_i \mathbf{b})$ and $G(\mathbf{b}')$ where $\varphi_k(\mathbf{b}') \leqslant \varphi_k(\mathbf{b})$ for all $k \in I$. Iterating this, we get that $0 = e_i e_j G(\tilde{e}_i \mathbf{b})$ is a linear combination of $G(\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b})$ and terms $G(\mathbf{b}')$. It is straightforward (using Eq. (3.1)) to show the restrictions on $\mathbf{b}'$ can only be satisfied if $\varepsilon_i(\tilde{e}_j \tilde{e}_i \mathbf{b}) \leqslant -a_{ij} - 1 = 0$. But this forces $\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b} = \mathbf{0}$. In the case there are no such $\mathbf{b}'$, we then get $G(\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b}) = 0$, so again $\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b} = \mathbf{0}$.

(3) The conditions on $\mathbf{b}$ give us $e_j G(\mathbf{b}) = 0$, $e_i G(\mathbf{b}) = G(\tilde{e}_i \mathbf{b})$, $e_i G(\tilde{e}_i \mathbf{b}) = 0$, and $e_j e_i G(\mathbf{b}) = G(\tilde{e}_j \tilde{e}_i \mathbf{b})$. The Serre relations imply $0 = e_i^{(2)} e_j e_i G(\mathbf{b}) = e_i^{(2)} G(\tilde{e}_j \tilde{e}_i \mathbf{b})$ which implies $\tilde{e}_i^2 \tilde{e}_j \tilde{e}_i \mathbf{b} = \mathbf{0}$. (Further $e_i e_j e_i G(\mathbf{b}) = G(\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b})$.) For the second case, we get $0 = e_i^{(3)} e_j G(\mathbf{b}) = e_i^{(3)} G(\tilde{e}_j \mathbf{b})$ so that $\tilde{e}_i^3 \tilde{e}_j \mathbf{b} = \mathbf{0}$.

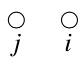
For the final statement, the proof of the first implication follows immediately from Eq. (3.1). For the second statement, as $a_{ji} = -1$, it follows $e_j e_i e_j e_i G(\mathbf{b}) = e_j e_i e_j G(\tilde{e}_i \mathbf{b}) = e_j^{(2)} e_i G(\tilde{e}_i \mathbf{b}) + e_i e_j^{(2)} G(\tilde{e}_i \mathbf{b}) = 0$. So $0 = e_j e_i e_j G(\tilde{e}_i \mathbf{b}) = e_j G(\tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b})$, yielding $\tilde{e}_j \tilde{e}_i \tilde{e}_j \tilde{e}_i \mathbf{b} = \mathbf{0}$. $\square$

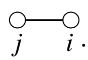
We remark that there are similar statements for $a_{ij} = -3, -4$, but they do not translate into interesting statements about singular nodes as the other cases do in Theorem 3.7 below.

Case (1) of Corollary 3.5 says that if $a_{ij} = 0$ and in the crystal graph we see

$$\bullet \xrightarrow{i} \mathbf{b}, \text{ then we do not see } \bullet \xrightarrow{j} \bullet \xrightarrow{i} \mathbf{b}.$$

Compare this with the fact that $a_{ij} = 0$ means that in the Dynkin diagram we see

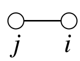


and not


.

Similarly, when $a_{ij} = -1$ and we see

$$\bullet \xrightarrow{j} \bullet \xrightarrow{i} \mathbf{b}, \text{ we do not see } \bullet \xrightarrow{i} \bullet \xrightarrow{j} \bullet \xrightarrow{i} \mathbf{b}.$$

Compare this to the fact that when $a_{ij} = -1$ we see


in the Dynkin diagram but

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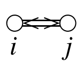

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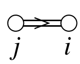
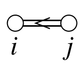
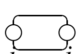
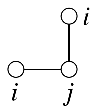


nor


which we should associate to


in the former case, and to the folding of



in the latter cases. In Theorem 3.7 below, we shall see that requiring certain singularity conditions on nodes forces the colors on their in-arrows to behave as a directed "path" or walk would on the Dynkin diagram, as suggested above. Choosing **a**, **b** with $\varepsilon(\mathbf{a}) = \Lambda_i$, $\varepsilon(\mathbf{b}) = \Lambda_j$ and where **b** is the parent of **a**, puts an "orientation" on the Dynkin diagram. As the Dynkin diagram's vertices correspond to arrows in the crystal, we really are making a statement about a graph *dual* to the Dynkin diagram. It turns out the correct notion of duality in this setting is exactly captured in an associated level 1 perfect crystal. These are listed in Appendix A.

Below, we recap, globally in Theorem 3.6 and case by case in Theorem 3.7, the consequences of Corollary 3.5 on all of the ancestors of a singular node **a** whose parent is also singular. We describe all walks on the crystal, from the highest weight node **v** to **a**. These walks are described exactly by walks on the level 1 perfect crystal $B^{1,1}$. These crystals are displayed in the body of the proof as well as in Appendix A. (In type $A_n^{(1)}$ we also need the perfect crystal $B^{n,1}$ obtained by reversing all arrows in $B^{1,1}$.)

A necessary, but not sufficient, condition for both a node and its parent to be singular is that it has the form $\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_k}\mathbf{v}$, where $\xrightarrow{i_1}\xrightarrow{i_2}\cdots\xrightarrow{i_k}$ is a consecutive sequence of arrows in $B^{1,1}$. (The theorem also describes which nodes of this form are not singular.) This means that we can not only give a case by case description of the node's ancestors, but also a global statement about the walks from **v** to **a**. The local nature of singularity means that the result in affine type follows from that in finite type (in small rank), and so we structure the statements and proofs of the following theorem accordingly.

Although it is possible to avoid listing the results type by type as in Theorem 3.7, we thought it would be useful for the reader to see the results in detail. The proof is *not* given type by type, but it is given in several cases, to account for whether $a_{ij} = a_{ji} = -1$, $a_{ij} = -1, a_{ji} = -2$, and so on. Hence each case covers multiple types, and also builds on the previous cases, inductively.

Theorem 3.6. *Let B be a highest weight crystal with highest weight node* **v** *of type A_n, $n \geqslant 1$; $A_n^{(1)}$, $n \geqslant 1$; $A_{2n}^{(2)}$, $n \geqslant 2$; $A_{2n}^{(2)\dagger}$, $n \geqslant 2$; $A_{2n-1}^{(2)}$, $n \geqslant 3$; B_n, $n \geqslant 2$; $B_n^{(1)}$, $n \geqslant 3$; C_n, $n \geqslant 2$; $C_n^{(1)}$, $n \geqslant 2$; D_n, $n \geqslant 4$; $D_n^{(1)}$, $n \geqslant 4$; $D_{n+1}^{(2)}$, $n \geqslant 2$. Suppose $\mathbf{a} \in B$ is a singular node with singular parent. Then*

$$\mathbf{a} = \tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_k}\mathbf{v},$$

only when $\xrightarrow{i_1}\xrightarrow{i_2}\cdots\xrightarrow{i_k}$ is a consecutive sequence of arrows in the level 1 perfect crystal $B^{1,1}$ (or $B^{n,1}$ in type A) of appropriate type, ignoring the 0-arrows in finite type. If the Dynkin diagram does not contain that of D_4 as a subdiagram, then this sequence is unique.

Observe that in types A_n, $A_n^{(1)}$, $C_n^{(1)}$, we can reinterpret the conclusion as saying all ancestors **c** have the level of $\varepsilon(\mathbf{c}) \leqslant 1$. In the other types, all ancestors **c** have the level of $\varepsilon(\mathbf{c}) \leqslant 2$.

For convenience, we introduce the following notation. Let J be an ordered tuple $J = (j_1, j_2, \ldots, j_k)$ with $j_\ell \in I$. Then we will write $\tilde{f}_J = \tilde{f}_{j_1}\tilde{f}_{j_2}\cdots\tilde{f}_{j_k}$. In the case **b** is a node in a crystal, $i \neq j$, and $\mathbf{a} = \tilde{f}_i\tilde{f}_j\mathbf{b} = \tilde{f}_j\tilde{f}_i\mathbf{b}$ we will also write this as $\mathbf{a} = \tilde{f}_J\mathbf{b}$ for $J = \binom{i}{j}$. This notation encodes the fact we can either let $J = (i, j)$ or (j, i).

Theorem 3.7.

(1) *Let B be a highest weight crystal of type A_n, $n \geqslant 1$, or of type $A_n^{(1)}$, $n \geqslant 1$.*

Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \tilde{e}_i\mathbf{a}$. Then all ancestors of $\mathbf{a}$ are singular. There is a unique walk (on the crystal as a directed graph) from the highest weight node $\mathbf{v} \in B$ to $\mathbf{a}$, given by $\mathbf{a} = \tilde{f}_J\mathbf{v}$ where $J = (i, i+1, i+2, \ldots, i+k)$ or $J = (i, i-1, \ldots, i-k)$ and entries are taken mod n. *(In future, we will write this as $J = (i, i \pm 1, \ldots, i \pm k)$.)*

(2) *Let B be a highest weight crystal of type C_n, $n \geqslant 2$.*

Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \tilde{e}_i\mathbf{a}$. Then all ancestors of $\mathbf{a}$ are singular. There is a unique walk from the highest weight node $\mathbf{v}$ to $\mathbf{a}$, given by $\mathbf{a} = \tilde{f}_J\mathbf{v}$, where $J = (i, i \pm 1, \ldots, i \pm k)$ or $J = (i, i+1, i+2, \ldots, n-1, n, n-1, n-2, \ldots)$.

(3) *Let B be a highest weight crystal of type B_n, $n \geqslant 2$.*

Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \tilde{e}_i\mathbf{a}$. Then all but one of the ancestors of $\mathbf{a}$ are singular. If there is a non-singular ancestor $\mathbf{c}$, it satisfies $\varepsilon(\mathbf{c}) = 2\Lambda_n$.

There is a unique walk from the highest weight node $\mathbf{v}$ to $\mathbf{a}$, given by $\mathbf{a} = \tilde{f}_J\mathbf{v}$, where $J = (i, i \pm 1, \ldots, i \pm k)$ or $J = (i, i+1, i+2, \ldots, n-1, n, n, n-1, n-2, \ldots)$.

(4) *Let B be a highest weight crystal of type D_n, $n \geqslant 4$.*

Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \tilde{e}_i\mathbf{a}$. Then all but one of the ancestors of $\mathbf{a}$ are singular. If there is a non-singular ancestor $\mathbf{c}$, it satisfies $\varepsilon(\mathbf{c}) = \Lambda_{n-1} + \Lambda_n$.

There are at most two walks from the highest weight node $\mathbf{v}$ to $\mathbf{a}$, given by $\mathbf{a} = \tilde{f}_J\mathbf{v}$, and the following possibilities. If $J = (i, i \pm 1, i \pm 2, \ldots)$ truncates before reaching $n-1$ or n, we get a unique walk. Or we can have $J = (i, i+1, \ldots, n-2, n-1, n, n-2, \ldots)$ which yields a second walk corresponding to $J' = (i, i+1, \ldots, n-2, n, n-1, n-2, \ldots)$. In such cases, we will combine J and J' into the abbreviated notation $J = (i, i+1, \ldots, n-2, {}^{\,n}_{n-1}, n-2, \ldots)$.

(5) *Let B be a highest weight crystal of type $C_n^{(1)}$, $n \geqslant 2$; $A_{2n}^{(2)}$, $n \geqslant 2$; $A_{2n}^{(2)\dagger}$, $n \geqslant 2$; or $D_{n+1}^{(2)}$, $n \geqslant 2$.*

Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \tilde{e}_i\mathbf{a}$. Then ancestors $\mathbf{c}$ of $\mathbf{a}$ are either singular or they can satisfy $\varepsilon(\mathbf{c}) = 2\Lambda_n$ in types $A_{2n}^{(2)\dagger}$, $D_{n+1}^{(2)}$; $\varepsilon(\mathbf{c}) = 2\Lambda_0$ in types $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$.

In all cases, there is a unique walk from the highest weight node $\mathbf{v}$ to $\mathbf{a}$, given by the following possibilities: $\mathbf{a} = \tilde{f}_J\mathbf{v}$, and $J = (i, i_2, \ldots, i_k)$ where $\xrightarrow{i} \xrightarrow{i_2} \cdots \xrightarrow{i_k}$ is a consecutive sequence of arrows in the level 1 perfect crystal $B^{1,1}$ of appropriate type.

(6) *Let B be a highest weight crystal of type $D_n^{(1)}$, $n \geqslant 4$; $B_n^{(1)}$, $n \geqslant 3$; or $A_{2n-1}^{(2)}$, $n \geqslant 3$.*

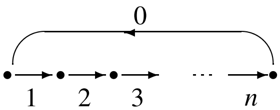
Suppose $\mathbf{a}, \mathbf{b} \in B$ are both singular nodes with $\mathbf{b} = \tilde{e}_i\mathbf{a}$. Then ancestors $\mathbf{c}$ of $\mathbf{a}$ are either singular or they can satisfy $\varepsilon(\mathbf{c}) = 2\Lambda_n$ in type $B_n^{(1)}$; $\varepsilon(\mathbf{c}) = \Lambda_{n-1} + \Lambda_n$, in type $D_n^{(1)}$; $\varepsilon(\mathbf{c}) = \Lambda_1 + \Lambda_0$ in types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$.

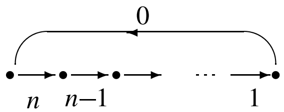
Walks from the highest weight node $\mathbf{v} \in B$ *to* $\mathbf{a}$, *described by the following* (*infinite*) *possibilities.* $\mathbf{a} = \tilde{f}_J \mathbf{v}$ *and* $J = (i, i_2, \ldots, i_k)$, *where* $\xrightarrow{i} \xrightarrow{i_2} \cdots \xrightarrow{i_k}$ *is a consecutive sequence of arrows in* $B^{1,1}$.

We remark that in cases not included above, such as exceptional types, or type $A_2^{(2)}$, that requiring a certain number of consecutive singular nodes either gives many possible complicated walks from the highest weight node or none at all. At the end of this paper we have a short discussion regarding type E_6.

Proof. (1) $[A_n, A_n^{(1)}]$ We will do case $A_1^{(1)}$ separately. We have $\varepsilon(\mathbf{a}) = \Lambda_i$, and either $\mathbf{b} = \mathbf{v}$ or $\varepsilon(\mathbf{b}) = \Lambda_j$ with j connected to i in the Dynkin diagram by Corollary 3.3. In this case $j = i \pm 1$, taking $j \bmod n$ if necessary. Applying this corollary again, $\tilde{e}_k(\tilde{e}_j \mathbf{b}) = \mathbf{0}$ unless $k = j \pm 1$. By case (2) of Corollary 3.5, $\mathbf{0} = \tilde{e}_i(\tilde{e}_j \tilde{e}_i \mathbf{a}) = \tilde{e}_i \tilde{e}_j \mathbf{b}$, so we must have $k = i \pm 2$ and either $\tilde{e}_j \mathbf{b} = \mathbf{v}$ or $\varepsilon(\tilde{e}_j \mathbf{b}) = \Lambda_k$. Hence we can inductively apply this argument to the pair $\mathbf{b}$ and $\tilde{e}_j \mathbf{b}$. As B is a highest weight crystal, this process must eventually terminate at $\tilde{e}_{i\pm m} \cdots \tilde{e}_{i\pm 1} \tilde{e}_i \mathbf{a} = \mathbf{v}$ which is equivalent to $\mathbf{a} = \tilde{f}_i \tilde{f}_{i\pm 1} \tilde{f}_{i\pm 2} \cdots \tilde{f}_{i\pm m} \mathbf{v}$.

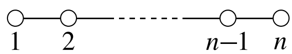
The above sequence of consecutively colored arrows exactly corresponds to a sequence of arrows on the following perfect crystals. In other words, $\mathbf{a} = \tilde{f}_J \mathbf{v}$, where $J = (i, i_2, i_3, \ldots)$ and $\xrightarrow{i} \xrightarrow{i_2} \cdots \xrightarrow{i_k}$ is a consecutive sequence of arrows in the level 1 perfect crystals $B^{1,1}$ or $B^{n,1}$ pictured below.

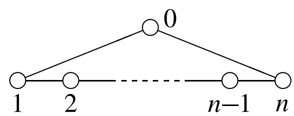




As we only care about the arrow labels, we omit the node labels that are usually also pictured in the crystals.

The reader should compare the above perfect crystals to the Dynkin diagrams

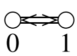
A_n: 

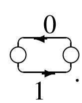
$A_n^{(1)}$: 

and consider the discussion below Corollary 3.5. Observe that arrows being consecutive in the perfect crystal correspond to vertices being adjacent in the Dynkin diagram.

We prove the case $A_1^{(1)}$ separately, as $a_{ij} = a_{ji} = -2$. As above, we necessarily have $\varepsilon(\mathbf{a}) = \Lambda_i$, $\varepsilon(\mathbf{b}) = \Lambda_j$ by Corollary 3.3. Let $\mathbf{c} = \tilde{e}_j \mathbf{b}$ and assume $\mathbf{c} \neq \mathbf{0}$. By Lemma 3.1, necessarily $\varepsilon(\mathbf{c}) = m\Lambda_i$. The Serre relations give us $e_i^{(2)} e_j e_i = e_i^{(3)} e_j + e_i e_j e_i^{(2)} - e_j e_i^{(3)}$, and by Theorem 3.4 each of the three right-hand side terms kills $G(\mathbf{a})$. Thus $0 = e_i^{(2)} e_j e_i G(\mathbf{a}) = e_i^{(2)} e_j G(\tilde{e}_i \mathbf{a}) = e_i^{(2)} G(\tilde{e}_j \tilde{e}_i \mathbf{a})$. Hence, again by Theorem 3.4, $\mathbf{0} = \tilde{e}_i^2 \tilde{e}_j \tilde{e}_i \mathbf{a} = \tilde{e}_i^2 \mathbf{c}$, showing $m < 2$ and so $\varepsilon(\mathbf{c}) = \Lambda_i$. In other words, $\mathbf{c}$ is singular. Now we proceed with a similar inductive argument as that used above.

Again, the reader can compare this statement to tracing a path on the Dynkin diagram

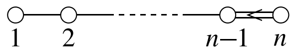
$A_1^{(1)}$: 

which is captured in the perfect crystal 
.

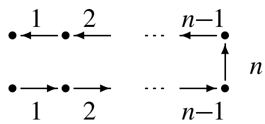
In all of these cases, note that all ancestors $\mathbf{c}$ of $\mathbf{a}$ have the level of $\varepsilon(\mathbf{c})$ less than or equal to 1.

(2) $[C_n]$ This proof is similar to that of case (1). We need only consider the case that $\mathbf{a}$ and $\mathbf{b} = \tilde{e}_{n-1}\mathbf{a}$ are singular with $\varepsilon(\mathbf{a}) = \Lambda_{n-1}, \varepsilon(\mathbf{b}) = \Lambda_n$. Let $\mathbf{c} = \tilde{e}_n\mathbf{b}$. We claim either $\mathbf{c} = \mathbf{v}$ or $\varepsilon(\mathbf{c}) = \Lambda_{n-1}$. By Corollary 3.3, we know $\tilde{e}_k\mathbf{c} = \mathbf{0}$ unless $k = n-1$. Because $a_{n-1,n} = -2$, by case (3) of Corollary 3.5, we know that $\tilde{e}_{n-1}^2\tilde{e}_n\tilde{e}_{n-1}\mathbf{a} = \mathbf{0}$. This gives the claim. Now the induction proceeds just as in type A.

Again, we draw the Dynkin diagram

C_n: 

and show the perfect crystal of type $C_n^{(1)}$ with the 0-arrow removed, which is suggestive of picturing the double arrow as a folding. (Note that we recover the same graph reversing orientation of all arrows.)



Note that the conclusions (a), (b) can also be expressed as $\mathbf{a} = \tilde{f}_i\tilde{f}_{i\pm 1}\cdots\tilde{f}_{i\pm k}\mathbf{v}$, so long as subscripts are taken mod $2n$, and one sets $\tilde{f}_{n+m} := \tilde{f}_{n-m}$ for $0 < m < n$.

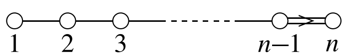
(3) $[B_n]$ We need only consider the case that $\mathbf{a}$ and $\mathbf{b} = \tilde{e}_{n-2}\mathbf{a}$ are singular with $\varepsilon(\mathbf{a}) = \Lambda_{n-2}, \varepsilon(\mathbf{b}) = \Lambda_{n-1}$. Otherwise it reduces to case (1). Let $\mathbf{c} = \tilde{e}_{n-1}\mathbf{b}$. We claim either $\mathbf{c} = \mathbf{v}$; $\varepsilon(\mathbf{c}) = \Lambda_n$ in which case $\tilde{e}_n\mathbf{c} = \mathbf{v}$; or $\varepsilon(\mathbf{c}) = 2\Lambda_n$, in which case $\mathbf{c}$ is *not* singular, but both $\tilde{e}_n\mathbf{c}$ and $\tilde{e}_n^2\mathbf{c}$ are singular, and $\tilde{e}_{n-1}\tilde{e}_n^2\mathbf{c}$ is either singular or $\mathbf{0}$.

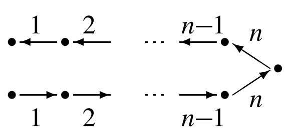
By Corollary 3.3, we know $\tilde{e}_k\mathbf{c} = \mathbf{0}$ unless $k = n$ or $n-2$. But case (1) of this theorem rules out the latter. Because $a_{n,n-1} = -2$, by case (3) of Corollary 3.5 we know $\tilde{e}_n^3\tilde{e}_{n-1}\mathbf{b} = \mathbf{0}$, showing the first part of the claim. Now suppose $\varepsilon(\mathbf{c}) = \Lambda_n$. That means $\tilde{e}_n^2\mathbf{c} = \mathbf{0}$. Further, $\tilde{e}_{n-1}\tilde{e}_n\mathbf{c} = \tilde{e}_{n-1}\tilde{e}_n\tilde{e}_{n-1}\mathbf{b} = \mathbf{0}$ by case (2) of Corollary 3.5 as $a_{n-1,n} = -1$. By Corollary 3.3, $\tilde{e}_k\tilde{e}_n\mathbf{c} = \mathbf{0}$ for all $k \neq n-1$, showing $\tilde{e}_n\mathbf{c} = \mathbf{v}$ as the crystal B has a unique highest weight node.

Next suppose $\varepsilon(\mathbf{c}) = 2\Lambda_n$. In particular, notice that $\mathbf{c}$ is not singular. For $k \neq n-1, n$, we know $0 = \varepsilon_k(\mathbf{c}) = \varepsilon_k(\tilde{e}_n\mathbf{c}) = \varepsilon_k(\tilde{e}_n^2\mathbf{c})$ by Lemma 3.1. As above, we still have $\tilde{e}_{n-1}\tilde{e}_n\mathbf{c} = \mathbf{0}$, so that $\tilde{e}_n\mathbf{c}$ is singular.

To show $\tilde{e}_n^2\mathbf{c}$ is singular, we need only show $\tilde{e}_{n-1}^2\tilde{e}_n^2\mathbf{c} = \mathbf{0}$. From the Serre relations, we know $(e_{n-1}^{(2)}e_n - e_{n-1}e_ne_{n-1} + e_ne_{n-1}^{(2)})G(\mathbf{b}) = 0$ and also $(e_{n-1}^{(2)}e_n - e_{n-1}e_ne_{n-1} + e_ne_{n-1}^{(2)})e_nG(\mathbf{c}) = 0$. From the first equation and the fact $\varepsilon(\mathbf{b}) = \Lambda_{n-1}$, we get that $0 = e_{n-1}e_ne_{n-1}G(\mathbf{b}) = e_{n-1}e_nG(\tilde{e}_{n-1}\mathbf{b}) = e_{n-1}e_nG(\mathbf{c})$. Hence we also see $0 = e_{n-1}^{(2)}e_nG(\mathbf{c})$. Thus from the second equation, we deduce $0 = e_{n-1}^{(2)}e_ne_nG(\mathbf{c}) = [2]e_{n-1}^{(2)}e_n^{(2)}G(\mathbf{c}) = [2]e_{n-1}^{(2)}G(\tilde{e}_n^2\mathbf{c})$. Hence $\tilde{e}_{n-1}^2\tilde{e}_n^2\mathbf{c} = \mathbf{0}$.

We again show the Dynkin diagram, along with the perfect crystal with 0-arrows removed.

B_n: 

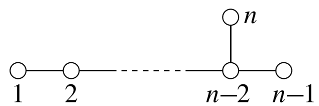


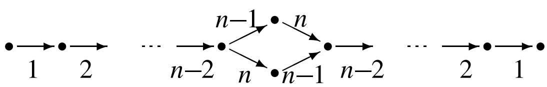
(4) $[D_n]$ For type D_n, we need only consider the case that $\mathbf{a}$ and $\mathbf{b} = \tilde{e}_{n-3}\mathbf{a}$ are singular with $\varepsilon(\mathbf{a}) = \Lambda_{n-3}$, $\varepsilon(\mathbf{b}) = \Lambda_{n-2}$. Let $\mathbf{c} = \tilde{e}_{n-2}\mathbf{b}$. By Corollaries 3.3 and 3.5, $\varepsilon(\mathbf{c}) = m_0\Lambda_n + m_1\Lambda_{n-1}$ with $0 \leqslant m_\ell \leqslant 1$. Let $m = m_0 + m_1$. If $m = 0$, then $\mathbf{c} = \mathbf{v}$ so we are done. If $m = 1 = m_\ell$, then $\tilde{e}_{n-\ell}\mathbf{c} = \mathbf{v}$ and again we are done. Otherwise, let $\mathbf{d} = \tilde{e}_n\tilde{e}_{n-1}\mathbf{c}$. Observe $\varepsilon_n(\tilde{e}_{n-1}\mathbf{c}) = \varepsilon_n(\mathbf{c}) = 1$ and $\varepsilon_{n-1}(\tilde{e}_n\mathbf{c}) = 1$ by Lemma 3.1. Thus $G(\tilde{e}_n\tilde{e}_{n-1}\mathbf{c}) = e_n e_{n-1}G(\mathbf{c}) = e_{n-1}e_n G(\mathbf{c}) = G(\tilde{e}_{n-1}\tilde{e}_n\mathbf{c})$, so that $\mathbf{d} = \tilde{e}_{n-1}\tilde{e}_n\mathbf{c}$ as well. If $k \neq n-2$, then $\varepsilon_k(\mathbf{d}) = \varepsilon_k(\mathbf{c}) = 0$. We can apply Theorem 3.4 to see $\tilde{e}_{n-2}^2\mathbf{d} = \mathbf{0}$, and so $\mathbf{d}$ is singular.

Standard arguments show either $\mathbf{d} = \mathbf{v}$ or $\tilde{e}_{n-2}\mathbf{d}$ is also singular. And then this reduces to case (1).

We again show the Dynkin diagram and the perfect crystal with 0-arrows removed.

D_n:

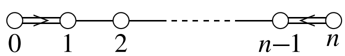


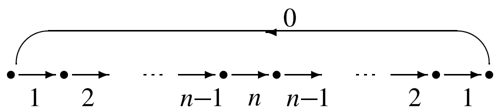


(5) $[C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)}]$ The local nature of singularity allows us to apply the results from cases (1)–(3) to these types (sometimes reindexing i for $n-i$ when encountering Λ_0).

We list the perfect crystals, and for completeness, the possible walks from $\mathbf{v}$ to $\mathbf{a}$.

$C_n^{(1)}$ $(n \geqslant 2)$,





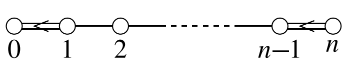
$\mathbf{a} = \tilde{f}_J\mathbf{v}$ where

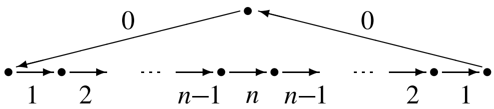
$$J = (i, i+1, \ldots, n-1, n, n-1, \ldots, 1, 0, 1, \ldots, n, \ldots)$$

or

$$J = (i, i-1, \ldots, 1, 0, 1, \ldots, n-1, n, n-1, \ldots, 0, \ldots, n, \ldots).$$

$A_{2n}^{(2)}$ $(n \geqslant 2)$,





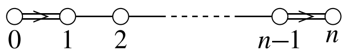
$\mathbf{a} = \tilde{f}_J \mathbf{v}$ where

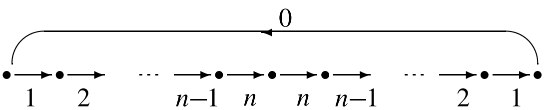
$$J = (i, i+1, \dots, n-1, n, n-1, \dots, 1, 0, 0, 1, \dots, n, \dots, 0, 0 \dots)$$

or

$$J = (i, i-1, \dots, 2, 1, 0, 0, 1, \dots, n-1, n, n-1, \dots, 0, 0, \dots, n, \dots).$$

$A_{2n}^{(2)\dagger}$ $(n \geqslant 2)$,





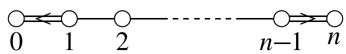
$\mathbf{a} = \tilde{f}_J \mathbf{v}$ where

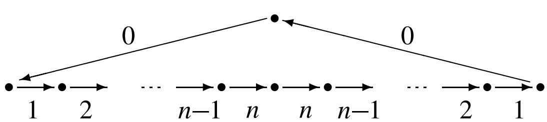
$$J = (i, i+1, \dots, n-1, n, n, n-1, \dots, 1, 0, 1, \dots, n, n, \dots, 0 \dots)$$

or

$$J = (i, i-1, \dots, 1, 0, 1, \dots, n-1, n, n, n-1, \dots, 0, \dots, n, n, \dots).$$

$D_{n+1}^{(2)}$ $(n \geqslant 2)$,





$\mathbf{a} = \tilde{f}_J \mathbf{v}$ where

$$J = (i, i+1, \dots, n-1, n, n, n-1, \dots, 1, 0, 0, 1, \dots, n, n, \dots, 0, 0 \dots)$$

or

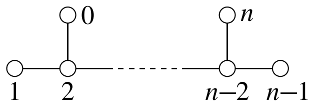
$$J = (i, i-1, \dots, 1, 0, 0, 1, \dots, n-1, n, n, n-1, \dots, 0, 0 \dots, n, n, \dots).$$

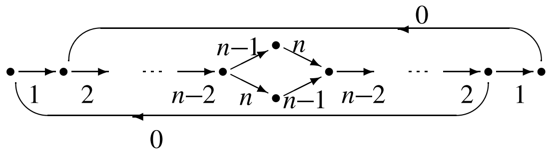
(6) $[D_n^{(1)}, A_{2n-1}^{(2)}, B_n^{(1)}]$ As above, we may apply the results from cases (1)–(4) to these types (with appropriate reindexing).

We list the perfect crystals, and for completeness, the possible walks from $\mathbf{v}$ to $\mathbf{a}$.

Below we again use the notation $\tilde{f}_J$ with $J = \binom{n}{n-1}$ to stand for either $\tilde{f}_{n-1}\tilde{f}_n$ or $\tilde{f}_n\tilde{f}_{n-1}$ in types D_n, $D_n^{(1)}$ and $\tilde{f}_J$ with $J = \binom{0}{1}$ to stand for either $\tilde{f}_0\tilde{f}_1$ or $\tilde{f}_1\tilde{f}_0$ in types $D_n^{(1)}$, $B_n^{(1)}$.

$D_n^{(1)}$ $(n \geqslant 4)$,





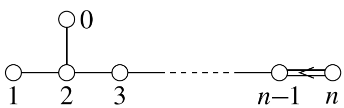
$\mathbf{a} = \tilde{f}_J\mathbf{v}$ where

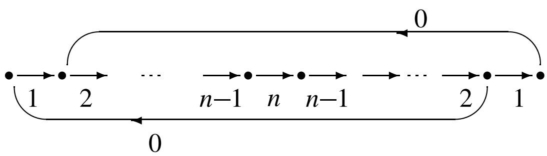
$$J = \left(i, i+1, \ldots, n-2, {}^{\;n}_{n-1}, n-2, \ldots, 2, {}^{0}_{1}, 2, \ldots, {}^{\;n}_{n-1}, \ldots, {}^{0}_{1}, \ldots\right)$$

or

$$J = \left(i, i-1, \ldots, 2, {}^{0}_{1}, 2, \ldots, n-2, {}^{\;n}_{n-1}, n-2, \ldots, {}^{0}_{1}, \ldots, {}^{\;n}_{n-1}, \ldots\right).$$

$A_{2n-1}^{(2)}$ $(n \geqslant 3)$,





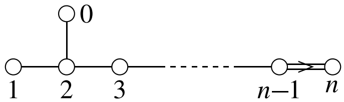
$\mathbf{a} = \tilde{f}_J\mathbf{v}$ where

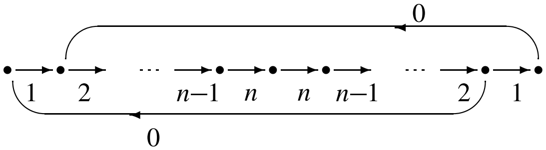
$$J = \left(i, i+1, \ldots, n-2, n-1, n, n-1, \ldots, 2, {}^{0}_{1}, 2, \ldots, n, \ldots, {}^{0}_{1}, \ldots\right)$$

or

$$J = \left(i, i-1, \ldots, 2, {}^{0}_{1}, 2, \ldots, n-2, n-1, n, n-1, \ldots, {}^{0}_{1}, \ldots, n, \ldots\right).$$

$B_n^{(1)}$ $(n \geqslant 3)$,





$\mathbf{a} = \tilde{f}_J \mathbf{v}$ where

$$J = \left(i, i+1, \ldots, n-1, n, n, n-1, \ldots, 3, 2, \genfrac{}{}{0pt}{}{0}{1}, 2, \ldots, n, n, \ldots, \genfrac{}{}{0pt}{}{0}{1}, \ldots\right)$$

or

$$J = \left(i, i-1, \ldots, 2, \genfrac{}{}{0pt}{}{0}{1}, 2, \ldots, n-1, n, n, n-1, \ldots, \genfrac{}{}{0pt}{}{0}{1}, \ldots, n, n, \ldots\right). \qquad \square$$

4. Existence

The above theorems consist of several "uniqueness" statements. The corresponding existence statements also hold.

In Theorems 3.6 and 3.7, we described all sequences $i_1, i_2, \ldots, i_k$ such that $\mathbf{a} = \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_k} \mathbf{v}$, where $\mathbf{v}$ is the highest weight node and we required $\mathbf{a}$ to be singular with singular parent. These possible sequences corresponded to walks $\xrightarrow{i_1} \xrightarrow{i_2} \cdots \xrightarrow{i_k}$ on a perfect crystal. Below we will exhibit a highest weight crystal (one of level 1 or level 2 suffices) and such a node $\mathbf{a}$ for every such walk (excluding of course walks where $a_{i_1,i_2} \geqslant 0$, as in that case $\mathbf{a}$ would not be singular).

We recall that the tensor product of crystals $B_2 \otimes B_1$ is defined by the nodes being the Cartesian product of the nodes of B_2 and B_1, $\mathrm{wt}(\mathbf{b}_2 \otimes \mathbf{b}_1) = \mathrm{wt}(\mathbf{b}_2) + \mathrm{wt}(\mathbf{b}_1)$, and arrows are described by the following rule

$$\tilde{e}_i(b_2 \otimes b_1) = \begin{cases} \tilde{e}_i b_2 \otimes b_1 & \text{if } \varphi_i(b_2) \geqslant \varepsilon_i(b_1), \\ b_2 \otimes \tilde{e}_i b_1 & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1). \end{cases}$$

Consequently

$$\varepsilon_i(b_2 \otimes b_1) = \varepsilon_i(b_2) + \max\bigl\{0, \varepsilon_i(b_1) - \varphi_i(b_2)\bigr\}, \tag{4.1}$$

$$\varphi_i(b_2 \otimes b_1) = \varphi_i(b_1) + \max\bigl\{0, \varphi_i(b_2) - \varepsilon_i(b_1)\bigr\}. \tag{4.2}$$

We recall the following theorem.

Theorem 4.1. *(See [5,6].) Let λ be a dominant integral weight of level k, and B be a perfect crystal of level ℓ, and suppose $k \geqslant \ell$. Then*

$$B(\lambda) \otimes B \simeq \bigoplus_{\mathbf{b} \in B^{\leqslant \lambda}} B\bigl(\lambda + \mathrm{wt}(\mathbf{b})\bigr)$$

where $B^{\leqslant \lambda} = \{\mathbf{b} \in B \mid \varepsilon_i(\mathbf{b}) \leqslant \langle h_i, \lambda \rangle\ \forall i\}$.

We set

$$\psi_k^{\lambda,\mu} : B(\mu) \to B(\lambda) \otimes \left(B^{1,1}\right)^{\otimes k}$$

to be the embedding dictated by the above theorem, when it is defined. Observe $(\psi_{k'}^{\lambda,\nu} \otimes \mathrm{id}^{\otimes k}) \circ \psi_k^{\nu,\mu} = \psi_{k'+k}^{\lambda+\mu}$.

We refer the reader to Appendix A for a list of the level 1 perfect crystals $B^{1,1}$ (including $B^{n,1}$ in type A). There is a standard way of labeling the nodes, but it will be convenient here to ignore that convention, so we have omitted that labeling in Appendix A.

Let $\xrightarrow{i_1} \xrightarrow{i_2} \cdots \xrightarrow{i_k}$ be a walk on $B^{1,1}$ (or $B^{n,1}$ in type A). Let

$$m = \left|\{r \mid a_{i_r,i_{r+1}} \geqslant 0,\ 1 \leqslant r < k\}\right| \tag{4.3}$$

and let

$$\boxed{\tilde{\imath}_1} \otimes \boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k} \in \left(B^{1,1}\right)^{\otimes (k-m)}$$

be such that $\boxed{\tilde{\imath}_r}$ is the node

$$\xrightarrow{i_r} \boxed{\tilde{\imath}_r} \xrightarrow{i_{r+1}}$$

with an i_r-colored arrow going in and i_{r+1} going out *if* $a_{i_r,i_{r+1}} < 0$. So long as $k > 1$, these nodes are well-defined and this also uniquely determines $\boxed{\tilde{\imath}_k}$. Observe that in the case $a_{i_1,i_2} \geqslant 0$, the node we describe is actually then $\boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}$. Also note the labeling very much depends on the walk, and that one node can receive many different labels.

Lemma 4.2. *Let* $\boxed{\tilde{\imath}_1} \otimes \boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k} \in (B^{1,1})^{\otimes (k-m)}$ *be as above. For all* $i \in I$,

(1) $\varepsilon_i(\boxed{\tilde{\imath}_1} \otimes \boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}) = \varepsilon_i(\boxed{\tilde{\imath}_1})$,

(2) $\tilde{e}_i(\boxed{\tilde{\imath}_1} \otimes \boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}) = \tilde{e}_i(\boxed{\tilde{\imath}_1}) \otimes \boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}$,

(3) $\varphi_i(\boxed{\tilde{\imath}_1} \otimes \boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}) = \varphi_i(\boxed{\tilde{\imath}_k})$.

Proof. We induct on k. When $k = 1$ this is immediate. Recall from (4.1), that $\varepsilon_i(\boxed{\tilde{\imath}_1} \otimes \boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}) = \varepsilon_i(\boxed{\tilde{\imath}_1}) + \max\{0, \varepsilon_i(\boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}) - \varphi_i(\boxed{\tilde{\imath}_1})\}$. Then, by the inductive hypothesis, we have $\varepsilon_i(\boxed{\tilde{\imath}_2} \otimes \cdots \otimes \boxed{\tilde{\imath}_k}) = \varepsilon_i(\boxed{\tilde{\imath}_2})$ (by which we mean the leftmost node in case $a_{i_2,i_3} \geqslant 0$). If $\varepsilon_i(\boxed{\tilde{\imath}_2}) = 0$, we are done. If $\varepsilon_i(\boxed{\tilde{\imath}_2}) \neq 0$, we will show $\varepsilon_i(\boxed{\tilde{\imath}_2}) - \varphi_i(\boxed{\tilde{\imath}_1}) \leqslant 0$.

Consider the following possibilities. First, $i = i_2$ and $\varepsilon_{i_2}(\boxed{\tilde{\imath}_2}) = 1$. As $\varphi_{i_2}(\boxed{\tilde{\imath}_1}) \geqslant 1$, we are done. Second, suppose $i = i_2$ and $\varepsilon_{i_2}(\boxed{\tilde{\imath}_2}) > 1$. In fact, because we assume $\boxed{\tilde{\imath}_1}$ contributes to the tensor and $\xrightarrow{i_2}$ joins $\boxed{\tilde{\imath}_1}$ to $\boxed{\tilde{\imath}_2}$, this cannot happen. It would mean $\boxed{\tilde{\imath}_2}$ does not contribute, and the "leftmost" node we refer to above is in fact $\boxed{\tilde{\imath}_3}$. We have

$$\xrightarrow{i_1} \boxed{\tilde{\imath}_1} \xrightarrow{i_2} \cdot \xrightarrow{i_3 = i_2} \boxed{\tilde{\imath}_3},$$

as $a_{i_2,i_3} = 2 \geqslant 0$, so that $\varepsilon_{i_2}(\widehat{\boxed{\tilde\imath_2}} \otimes \boxed{\tilde\imath_3} \otimes \cdots \boxed{\tilde\imath_k}) = \varepsilon_{i_2}(\boxed{\tilde\imath_3}) = \varphi_{i_2}(\boxed{\tilde\imath_1}) = 2$.

Third, suppose $i \neq i_2$. Then we must have

$$\begin{tikzcd} & \bullet \arrow[dr, "i_2"] & \\ \boxed{\tilde\imath_1} \arrow[ur, "i"] \arrow[dr, "i_2"'] & & \boxed{\tilde\imath_3} \\ & \bullet \arrow[ur, "i"'] & \end{tikzcd}$$

and $\varepsilon_i(\boxed{\tilde\imath_3}) = \varphi_i(\boxed{\tilde\imath_1}) = 1$. Again, $\boxed{\tilde\imath_2}$ does not contribute.

Computing φ_i is similar. The above conclusions come from examining all $B^{1,1}$ and from our definition of the node in $(B^{1,1})^{\otimes(k-m)}$ that our walk specifies.

The rule for computing $\tilde e_i$ of a tensor product gives us the second statement. $\square$

Write $\mathbf{v}_\lambda \in B(\lambda)$ for the highest weight node.

Proposition 4.3. *Let $\xrightarrow{i_1} \xrightarrow{i_2} \cdots \xrightarrow{i_k}$ be a walk on $B^{1,1}$ (or $B^{n,1}$ in type A). Let $\boxed{\tilde\imath_0}$ be the node such that $\boxed{\tilde\imath_0} \xrightarrow{i_1} \boxed{\tilde\imath_1}$, and let $\lambda = \varepsilon(\boxed{\tilde\imath_0})$. Let $\mu = \varphi(\boxed{\tilde\imath_{k-1}})$. Then*

(1) $\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_0} \otimes \boxed{\tilde\imath_1} \otimes \cdots \otimes \boxed{\tilde\imath_{k-1}} = \psi^{\lambda,\mu}_{k-m}(\mathbf{v}_\mu)$,

(2) $\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_1} \otimes \cdots \otimes \boxed{\tilde\imath_k} = \psi^{\lambda,\mu}_{k-m}(\mathbf{a})$, *where* $\mathbf{a} = \tilde f_{i_1} \tilde f_{i_2} \cdots \tilde f_{i_k} \mathbf{v}_\mu$,

(3) *in particular* $\mathbf{a} \neq \mathbf{0}$, *and if* $\boxed{\tilde\imath_1} \otimes \cdots \otimes \boxed{\tilde\imath_k}$ *is singular (with singular parent) so is* $\mathbf{a}$.

Proof. (1) From (4.1), $\varepsilon_i(\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_0} \otimes \boxed{\tilde\imath_1} \otimes \cdots \otimes \boxed{\tilde\imath_{k-1}}) = \varepsilon_i(\mathbf{v}_\lambda) + \max\{0, \varepsilon_i(\boxed{\tilde\imath_0} \otimes \cdots \otimes \boxed{\tilde\imath_{k-1}}) - \varphi_i(\mathbf{v}_\lambda)\} = 0 + \max\{0, \varepsilon_i(\boxed{\tilde\imath_0}) - \varphi_i(\mathbf{v}_\lambda)\} = 0$ for all i by our choice of λ. Hence it is a highest weight node. Lemma 4.2 computes its weight is μ, so it must be the image of $\mathbf{v}_\mu$. Notice μ is of level 1 or 2.

(2) We only need show

$$\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_1} \otimes \cdots \otimes \boxed{\tilde\imath_k} = \tilde f_{i_1} \tilde f_{i_2} \cdots \tilde f_{i_k}(\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_0} \otimes \boxed{\tilde\imath_1} \otimes \cdots \otimes \boxed{\tilde\imath_{k-1}}).$$

We will induct on k.

In the case $k = 1$, $\tilde e_i(\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_1}) = \mathbf{0}$ if $i \neq i_1$, and then $\tilde e_{i_1}(\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_1}) = \mathbf{v}_\lambda \otimes \tilde e_{i_1}\boxed{\tilde\imath_1} = \mathbf{v}_\lambda \otimes \boxed{\tilde\imath_0} = \psi^{\lambda,\Lambda_{i_1}}_1(\mathbf{v}_{\Lambda_{i_1}})$. So $\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_1}$ is singular with $\xrightarrow{i_1}$ describing the only walks from the appropriate highest weight node to it.

We compute, using the inductive hypothesis,

$$\begin{aligned} \psi^{\lambda,\mu}_{k-m}(\mathbf{a}) &= \tilde f_{i_1}\big(\psi^{\lambda,\Lambda_{i_1}}_1 \otimes \mathrm{id}^{\otimes(k-1-m)}\big) \circ \psi^{\Lambda_{i_1},\mu}_{k-1-m}(\tilde f_{i_2} \cdots \tilde f_{i_k} \mathbf{v}_\mu) \\ &= \tilde f_{i_1}\big(\psi^{\lambda,\Lambda_{i_1}}_1 \otimes \mathrm{id}^{\otimes(k-1-m)}\big)(\mathbf{v}_{\Lambda_{i_1}} \otimes \boxed{\tilde\imath_2} \otimes \cdots \otimes \boxed{\tilde\imath_k}) \\ &= \tilde f_{i_1}\big((\mathbf{v}_\lambda \otimes \boxed{\tilde\imath_0}) \otimes \boxed{\tilde\imath_2} \otimes \cdots \otimes \boxed{\tilde\imath_k}\big) \\ &= \mathbf{v}_\lambda \otimes \boxed{\tilde\imath_1} \otimes \cdots \otimes \boxed{\tilde\imath_k}. \end{aligned}$$

(3) This follows from Lemma 4.2, and that $\psi^{\lambda,\mu}_{k-m}$ is an embedding. Note that for $k \geqslant 3$, so long as $a_{i_1,i_2} < 0, a_{i_2,i_3} < 0$ the node will be singular with singular parent. $\square$

5. Representation-theoretic interpretation in type A

In this paper, we studied a singular node whose parent is also singular in highest weight crystals of finite and affine type. The papers [7,8] characterized all singular nodes in a level one highest weight crystal of type $A^{(1)}$ (and [9] for higher levels) by their behavior under tensor product of crystals, and they gave a representation-theoretic interpretation of these singular nodes as answering the Jantzen–Seitz problem. These nodes correspond to irreducible modules of the finite Hecke algebra H_n^{fin} of type A that remain irreducible on restriction from H_n^{fin} to H_{n-1}^{fin} (or for the Ariki–Koike (cyclotomic Hecke) algebras). One may then ask: which irreducible modules of H_n^{fin} remain irreducible on restriction to H_{n-2}^{fin}?

The dictionary between crystals and representations of Hecke algebras is as follows. In type $A_n^{(1)}$ (or A_∞), the nodes in highest weight crystals correspond to simple modules of cyclotomic (or affine) Hecke algebras of type A by [2,10]. In [2], it is shown that arrows in the crystal correspond to the simple submodules of restriction. (Further, the multiplicity of a simple submodule as a composition factor in the restriction is controlled by ε.) Hence a node being singular exactly means the restriction of the corresponding simple module is again simple.

Theorem 3.7 in type A was motivated by the following representation-theoretic fact, which addresses the question just posed, regarding restricting twice. If an irreducible module M of the *affine* Hecke algebra H_n of type A is irreducible on restriction to H_{n-2}, then M is one-dimensional and either a trivial or Steinberg (sign) module. In particular, its restriction to H_{n-k} is also irreducible. The dictionary described above says this Hecke-theoretic statement and case (1) of Theorem 3.7 are equivalent. However, a purely representation-theoretic proof is as straightforward as the crystal-theoretic proofs above.

Compare the representation-theoretic translation of the crystal-theoretic proof (given below) with the following direct proof communicated by Grojnowski.

Let H_n denote the affine Hecke algebra of type A. This algebra is defined over a field (or ring) $\mathbb{F}$ and depends on a parameter $q \in \mathbb{F}^\times$. It has generators

$$T_1, \dots, T_{n-1}, \quad X_1^{\pm 1}, \dots, X_n^{\pm 1}$$

and relations

$$\begin{aligned} &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad |i-j| > 1, \quad (T_i + 1)(T_i - q) = 0, \\ &X_i X_j = X_j X_i \quad \forall i, j, \quad T_i X_i T_i = q X_{i+1}, \quad T_i X_j = X_j T_i \quad \text{if } j \neq i, i+1. \end{aligned}$$

When we specialize $q = 1$, we recover the group algebra of the wreath product of $\mathbb{Z}$ with the symmetric group. In that case, T_i degenerates to the simple transposition $s_i = (i, i+1)$. We set the finite Hecke algebra H_n^{fin} of type A to be the algebra generated by just the $\{T_i \mid 1 \leqslant i < n\}$. We note this algebra can be realized as not only a subalgebra but as a quotient of the affine Hecke algebra.

Let M be an irreducible module of H_n, and suppose $\mathrm{Res}_{H_{n-2}}^{H_n} M$ is an irreducible H_{n-2}-module. In particular, the generator T_{n-1} commutes with H_{n-2} and so acts by a scalar on all of M, where that scalar is -1 or q, as $(T_{n-1} + 1)(T_{n-1} - q) = 0$. All of the T_i are conjugate in H_n, so all the T_i also act by that same scalar on all of M. In the case that scalar is q, M must have been a trivial module, and when it is -1 we have a Steinberg (sign) module. In particular, M is one-dimensional and $\mathrm{Res}_{H_k}^{H_n} M$ is irreducible for all $k \leqslant n$. (X_1 may act as any scalar on M, and then that determines the action of the X_i.) This argument is the correct explanation for the result, but it is unclear what its interpretation is in other types.

In contrast, here is the representation-theoretic version of the crystal-theoretic proof given in case (1) of Theorem 3.7.

We refer the reader to [2,11], or [10] for all the definitions (as it is not the main focus of this paper). Let M be an irreducible module of H_n. There are functors

$$e_i : \operatorname{Rep} H_n \to \operatorname{Rep} H_{n-1}$$

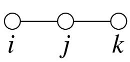
that satisfy $\bigoplus_i e_i M = \operatorname{Res}^{H_n}_{H_{n-1}} M$. Further, if $e_i^2 M = 0$ but $e_i M \neq 0$, then $e_i M$ is an irreducible H_{n-1}-module, and conversely. A node being singular corresponds to $\operatorname{Res}^{H_n}_{H_{n-1}} M$ being irreducible. Hence the hypotheses of case (1) of Theorem 3.7 correspond to the assumption that $\operatorname{Res}^{H_n}_{H_{n-1}} M = e_i M$ is irreducible, $e_{i+1} e_i M$ is also irreducible, and $e_j e_i M = 0$ for $j \neq i+1$. This implies $\operatorname{Res}^{H_n}_{H_{n-2}} M = e_{i+1} e_i M$ is irreducible. We want to conclude that $e_{i+2} e_{i+1} e_i M$ is also irreducible or zero (so we need only show $e_{i+2}^2 e_{i+1} e_i M = 0$, and that $e_j e_{i+1} e_i M = 0$ for $j \neq i+2$). These all follow from the fact, shown in [2,10,11] that the e_i satisfy the Serre relations of type A. (For ease of exposition, we omit the case where the parameter q appearing in the definition of H_n is a second root of unity, which implies that the T_i may not act semisimply, and corresponds to type $A_1^{(1)}$, where the Serre relations are of higher degree.) The proof here is very close to that of case (1) of Theorem 3.7 and (2) of Corollary 3.5, as they both rely on the Serre relations.

We also point out that this statement is obvious for the representation theory of the symmetric group in characteristic 0. Here, irreducible representations are indexed by partitions, and the branching rule says the restriction of an irreducible can be described by removing certain boxes from the partition. For the restriction of an irreducible module from S_n to S_{n-1} to be irreducible means its partition can have at most one removable box, and hence be a rectangle. But for that rectangle to share the same property, the original shape must have been a single row or column, hence our original representation was the trivial or sign module. We remark that the combinatorics in prime characteristic are appreciably different.

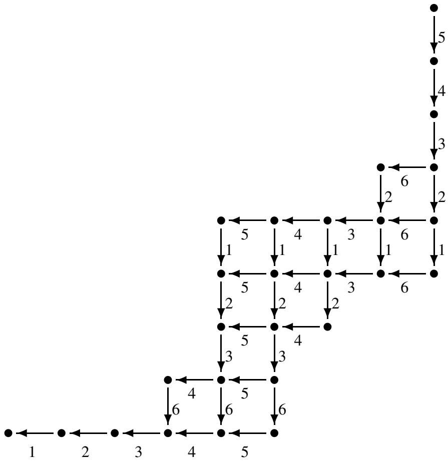
While for symmetric group modules in characteristic 0 this is a classical fact, it seemed a surprising statement for crystals: that two consecutive singular nodes could determine all of their ancestors, and that the perfect crystal $B^{1,1}$ controls all the paths between that node and the highest weight node.

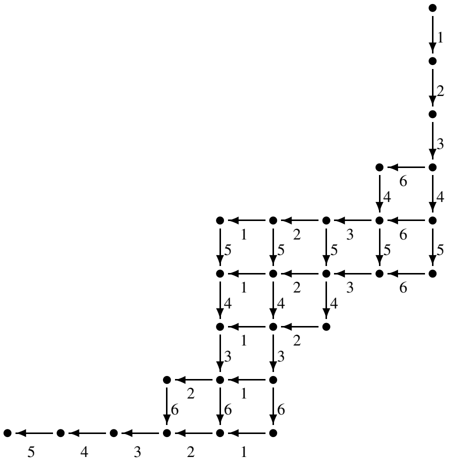
6. Exceptional types

Corollaries 3.2, 3.3, 3.5 say that in simply laced type, if $\varepsilon(\mathbf{a}) = \Lambda_i, \varepsilon(\tilde{e}_i \mathbf{a}) = \Lambda_j$, and $\varepsilon_k(\tilde{e}_j \tilde{e}_i \mathbf{a}) \neq 0$, then we see



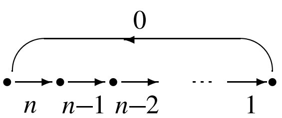
in the Dynkin diagram. In particular $k \neq i$. In classical types, the possible $\mathbf{v} \xrightarrow{i_k} \cdots \xrightarrow{i_2} \xrightarrow{i_1} \mathbf{a}$ are in correspondence with walks on $B^{(1,1)}$. In exceptional types, one can also describe a directed graph the corresponding walks must live on. The directed graph is dictated by Eq. (3.1) and case (2) of Corollary 3.5. They are very complicated to draw (planarly), so we only give pictures for E_6 below. Just as in type A, the two graphs below differ by reversing orientation of all arrows.



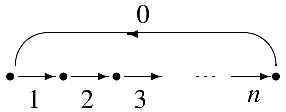


Appendix A

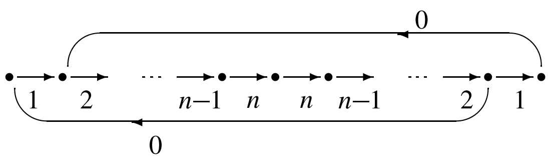
The crystal graphs $B^{1,1}$ are listed below.
We also need $B^{n-1,1}$ in type A:



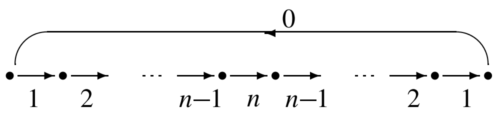
$A_n^{(1)}$:



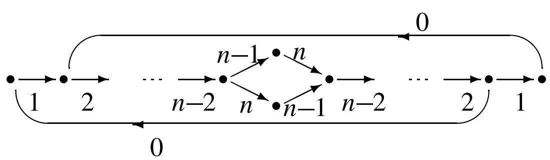
$B_n^{(1)}$:



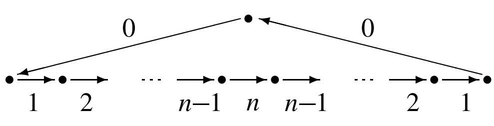
$C_n^{(1)}$:

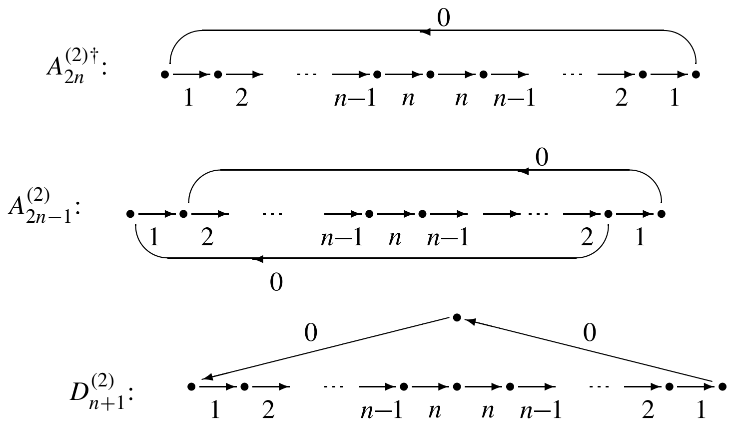


$D_n^{(1)}$:



$A_{2n}^{(2)}$:





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