Contemporary Mathematics

Irreducibility of affine Hecke algebra modules induced from Specht modules

Monica Vazirani

Contents

1. Introduction

This paper is a contribution to the extensive literature on affine Hecke algebras of type A and their modules, which began with the papers [**BZ**] and [**Z**]. In [**A**], [**A2**], [**AK**], [**AM**], [**BZ**], [**G**], [**G2**], [**Kv3**], [**LLT**], [**M**], [**V**], [**Z**] (to name only a few), the irreducible modules have been constructed and parameterized in various different ways as a distinguished composition factor of a module induced up from smaller Hecke algebras. This paper determines when an induced module is already irreducible for the construction described in [**V**]. Thus this paper is a natural sequel to [**V**]. Note that this question has been answered by [**LNT**] but that their methods use the theory of canonical bases, whereas the methods used here are elementary.

The notation used below in the introduction will be explained further in sections 2, 3, 4.

It was shown in [**G**] and [**A**],[**AM**] that the combinatorial data of the nodes of a suitable "crystal graph" parameterize the irreducible modules of the affine Hecke algebra of type A, denoted H_n, and also certain of its finite-dimensional quotients, the cyclotomic Hecke algebras, denoted $H_n^\lambda(q)$. In [**V**] we gave a direct construction of the irreducibles from that combinatorial data when the parameter q was not a

1991 *Mathematics Subject Classification.* Primary 20C08; Secondary 05E10.

The author was supported in part by an NSF Mathematical Sciences Postdoctoral Research Fellowship.

root of unity (and we also compared it to the direct construction of [**BZ**], [**Z**] and to the less direct but more powerful one of [**G**]).

That combinatorial data consists of "Kleshchev multipartitions" which are r-tuples of partitions $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ which satisfy

$$(*) \qquad \mu^{(t)}_{i_t - i_{t+1} + x} \leq \mu^{(t+1)}_x \qquad \text{for all } x \geq 1,\ 1 \leq t \leq r-1$$

where $i_1 \geq i_2 \geq \cdots \geq i_r$ have been fixed.

Those readers who are familiar with crystal graphs will recognize $(*)$ is equivalent to $\mu^{(r)} \otimes \cdots \otimes \mu^{(1)}$ being in the connected component of $\emptyset \otimes \cdots \otimes \emptyset$ in the tensor product of crystal graphs $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$, and that this component is $B(\lambda) = B(\Lambda_{i_r} + \cdots + \Lambda_{i_1})$. However, no knowledge of crystal graphs is necessary here.

By abuse of notation, let a single partition $\mu^{(j)}$ also denote the Specht module indexed by $\mu^{(j)}$, which is made into a module of the affine Hecke algebra via an evaluation homomorphism (depending on i_j) from H_n to the finite Hecke algebra of type A. Then we have denoted

$$\operatorname{Ind} \underline{\mu} = \operatorname{Ind} \mu^{(1)} \boxtimes \cdots \boxtimes \mu^{(r)}$$

and

$$N_{\underline{\mu}} = \operatorname{cosoc} \operatorname{Ind} \underline{\mu} = \operatorname{cosoc} \operatorname{Ind} \mu^{(1)} \boxtimes \cdots \boxtimes \mu^{(r)},$$

where the cosocle is the largest semisimple quotient. The $N_{\underline{\mu}}$ are distinct, irreducible, and range over all irreducible modules of $H^{\lambda}_n(q)$, $n \geq 0$, as the $\underline{\mu}$ range over all r-tuples of partitions satisfying $(*)$.

Sometimes the $\operatorname{Ind} \underline{\mu}$ are already irreducible, i.e. $N_{\underline{\mu}} = \operatorname{Ind} \underline{\mu}$. This paper describes for which $\underline{\mu}$ this holds.

In the main theorem (theorem 5.2) of this paper we prove that for a Kleshchev multipartition $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$, the induced module $\operatorname{Ind} \underline{\mu}$ is irreducible iff one cannot remove a border strip from the Young diagram of some $\mu^{(i)}$ and add it (with the same shape and filling) to the Young diagram of some $\mu^{(j)}$ with $i < j$. The fillings are determined by λ.

This condition is the same as that given in [**LNT**] for when "induction products of evaluation modules of type A affine Hecke algebras" are irreducible, although our wording is slightly different. The [**LNT**] result is stronger because it is a condition for any multipartition, not just a Kleshchev multipartition. We will only prove the irreducibility criterion for Kleshchev multipartitions. However the proofs here are elementary and combinatorial, and we avoid the machinery of canonical bases that [**LNT**] uses.

The proofs of results here require some familiarity with the work of [**G**], [**GV**], [**V**], [**V2**], and so several of their propositions and theorems are listed in sections 2.3, 2.4, 3,4, without proof. The reader familiar with the properties of Hecke algebras may skip to section 3 or 4. The main theorem is in section 5.2.

2. Notation and definitions

Let R be a field. If A is an R-algebra, we write $\operatorname{Rep} A$ for the category of left A-modules which are *finite dimensional* as R-modules. We recall that the socle of a module M, denoted $\operatorname{soc}(M)$, is the largest semisimple submodule of M, and that the cosocle of M, denoted $\operatorname{cosoc}(M)$, is its largest semisimple quotient.

If A and A' are two R-algebras with modules M and M' respectively, let $M \boxtimes M'$ denote the $A \otimes A'$-module which is isomorphic to $M \otimes_R M'$ as an R-module and has $A \otimes A'$ action given by

$$(a \otimes a') \cdot (m \otimes m') = am \otimes a'm'.$$

If R is an algebraically closed field and if M is an irreducible A-module and M' is an irreducible A'-module, then $M \boxtimes M'$ will be an irreducible representation of $A \otimes A'$, and all such are of this form.

2.1. Induction and Restriction. Recall that if $A \subset B$ are R-algebras such that B is finitely generated both as a left and right A-module, the exact functor of restriction

$$\mathrm{Res}_A^B : \mathrm{Rep}\, B \to \mathrm{Rep}\, A$$

has left and right adjoints, Ind and $\widehat{\mathrm{Ind}}$, defined by

$$\mathrm{Ind}_A^B : \mathrm{Rep}\, A \to \mathrm{Rep}\, B \qquad M \mapsto B \otimes_A M$$

$$\widehat{\mathrm{Ind}}_A^B : \mathrm{Rep}\, A \to \mathrm{Rep}\, B \qquad M \mapsto \mathrm{Hom}_A(B, M);$$

i.e.,

$$\mathrm{Hom}_B(\mathrm{Ind}\, M, N) = \mathrm{Hom}_A(M, \mathrm{Res}\, N) \quad \mathrm{Hom}_B(N, \widehat{\mathrm{Ind}} M) = \mathrm{Hom}_A(\mathrm{Res}\, N, M).$$

If B is a free A-module, then Ind and $\widehat{\mathrm{Ind}}$ are exact functors also. Further, if $A \subset B \subset C$ are inclusions of R-algebras, we have transitivity of induction and restriction:

$$\mathrm{Res}_A^B \,\mathrm{Res}_B^C = \mathrm{Res}_A^C, \quad \mathrm{Ind}_B^C \,\mathrm{Ind}_A^B = \mathrm{Ind}_A^C, \quad \widehat{\mathrm{Ind}}_B^C \widehat{\mathrm{Ind}}_A^B = \widehat{\mathrm{Ind}}_A^C.$$

2.2. H_n^{fin}, H_n and H_n^λ. Throughout the paper we fix an algebraically closed field R, and an invertible element $q \in R$, such that $q^\ell = 1$ implies $\ell = 0$. In this situation $\{q^i\}_{i \in \mathbb{Z}}$ is infinite and we say q is *generic*.

The *finite Hecke algebra*, H_n^{fin} is the R-algebra with generators

$$T_1, \dots, T_{n-1}$$

and relations

$$\text{(2.1)} \qquad \textit{braid relations} \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \ |i - j| > 1$$

$$\text{(2.2)} \ \textit{quadratic relations} \qquad (T_i + 1)(T_i - q) = 0.$$

The braid relations imply that if $w = s_{i_1} \cdots s_{i_k}$ and $\ell(w) = k$, then $T_{i_1} \cdots T_{i_k}$ depends only on $w \in S_n$. It is denoted T_w, and the $\{T_w \mid w \in S_n\}$ form a basis of H_n^{fin} over R.

The *affine Hecke algebra* H_n (or H_n^{aff}) is the R-algebra, which as an R-module is isomorphic to

$$H_n^{\mathrm{fin}} \otimes_R R[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$

The algebra structure is given by requiring that H_n^{fin} and $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be subalgebras, and that

$$T_i X_i T_i = q X_{i+1}. \tag{2.3}$$

This presentation of H_n is from [**BZ**].

Denote by $\mathrm{Rep}_q H_n$ the finite dimensional modules M for H_n such that the only eigenvalues of the X_j on M are powers of q. Let $\mathrm{Rep}_q = \bigoplus_{n \geq 0} \mathrm{Rep}_q H_n$.

REMARK 1. It follows from the computation in [**G**], section 6.2, that to understand $\operatorname{Rep} H_n$ it is enough to understand $\operatorname{Rep}_q H_n$.

The *cyclotomic Hecke algebra* or *Ariki-Koike algebra* H_n^λ is the quotient H_n/I_λ, where I_λ is the ideal generated by the polynomial in X_1: $\prod(X_1 - q^i)^{m_i}$, and $\lambda = \sum m_i \Lambda_i$ is a weight of $\mathfrak{gl}_\infty$ (where $\{\Lambda_i \mid i \in \mathbb{Z}\}$ denote the fundamental weights), [**AK**]. We may also write $\lambda = \sum_{k=1}^r \Lambda_{i_k}$ and so I_λ is generated by $\prod_{k=1}^r (X_1 - q^{i_k})$.

The motivation for viewing λ as a weight comes from theorems of [**G**] and [**A**],[**LLT**] about connections between H_n^λ, the crystal graph $B(\lambda)$, and the irreducible representation of $\mathfrak{gl}_\infty$ with highest weight λ. However, the reader need not know anything about $\mathfrak{gl}_\infty$, and simply think of λ as the collection $\{i_1, \ldots, i_r\}$.

REMARK 2. Observe that any irreducible module in Rep_q is an irreducible H_n^λ-module if we take λ large enough. Just let $\prod_{i=1}^r (X_1 - q^i)^{m_i}$ be the characteristic polynomial of X_1 acting on the module.

Conversely, we identify any irreducible H_n^λ-module as an irreducible H_n-module (on which $\prod(X_1 - q^i)^{m_i}$ vanishes) via the evaluation homomorphism

$$H_n \twoheadrightarrow H_n/I_\lambda.$$

The cyclotomic Hecke algebra is a deformation of $R[(\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n]$. When $q = 1$ and X_1 satisfies $X_1^r = 1$ (i.e. in place of the q^{i_k} we take r^{th} roots of unity), the cyclotomic Hecke algebra is isomorphic to the group algebra of $(\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$.

Given a sequence $P = (a_1, \ldots, a_k)$ of non-negative integers summing to n, write $H_P = H_{a_1} \otimes \cdots \otimes H_{a_k}$. We have an obvious embedding

$$H_P = H_{a_1} \otimes \cdots \otimes H_{a_k} \hookrightarrow H_n$$

which makes H_n a free H_P-module. Applying the remarks from section 2.1 we get exact functors $\operatorname{Res}, \operatorname{Ind}, \widehat{\operatorname{Ind}}$. When unambiguous from context, we just write Res_P^n or Res for $\operatorname{Res}_{H_P}^{H_n}$, and similarly for Ind to lighten notation. These functors depend on the order $(a_1, \ldots, a_k)$ and not just on the underlying set.

Throughout the paper when discussing H_n^λ-modules, we will really view them as H_n-modules upon which I_λ vanishes. This is especially crucial when applying the functors Ind and Res.

2.3. e_i, $\widetilde{e}_i$, and ε_i. Given an irreducible H_n-module M, consider the restriction $\operatorname{Res}_{n-1,1}^n M = \operatorname{Res}_{H_{n-1} \otimes H_1}^{H_n} M$. As $X_n - q^i \subseteq Z(H_{n-1} \otimes H_1)$, left multiplication by $(X_n - q^i)^m$ induces an endomorphism of $\operatorname{Res}_{H_{n-1} \otimes H_1}^{H_n} M$, and its kernel for $m \gg 0$ (we can take $m \le \dim M$) is the generalized eigenspace of $X_n - q^i$. Define $e_i M$ to be the restriction of that generalized eigenspace to H_{n-1}. Because restriction and taking generalized eigenspaces are exact functors, we have the following proposition.

PROPOSITION 1. *The functor* e_i *is an exact functor* $\operatorname{Rep}_q H_n \to \operatorname{Rep}_q H_{n-1}$ *and* $\operatorname{Res}_{H_{n-1}}^{H_n} M = \bigoplus_{i \in \mathbb{Z}} e_i M$.

Define

$$\varepsilon_i(M) = \max\{m \ge 0 \mid e_i^m M \ne 0\}.$$

We also define

$$\widetilde{e}_i M = \operatorname{soc} e_i M.$$

Observe $\varepsilon_i(M) = \max\{m \geq 0 \mid \widetilde{e}_i^m M \neq 0\}$ as well. We remind the reader that the socle of a module N, denoted $\mathrm{soc}(N)$, is the largest semisimple submodule of N, and that the cosocle of N, denoted $\mathrm{cosoc}(N)$, is its largest semisimple quotient.

Below and in the next section we collect useful results that we will need later. The first is the main theorem of [**GV**], and the second can be found there or in [**V**]. The rest are well-known and most can be found in [**B**], [**G**], [**GV**], [**Ka**], [**V2**].

Theorem 2.1 ([**GV**] Theorem B). *Let M be an irreducible H_n-module. Then $\widetilde{e}_i M$ is zero or is irreducible, and $\mathrm{cosoc}\, e_i M$ is isomorphic to $\widetilde{e}_i M$.*

Proposition 2. *Let M and L be irreducible H_n-modules. If $\widetilde{e}_i M \simeq \widetilde{e}_i L \neq 0$ then $M \simeq L$.*

Proposition 3. *Let M be an irreducible H_n-module and let $\varepsilon = \varepsilon_i(M)$. Then $e_i^\varepsilon M$ is the direct sum of $\varepsilon!$ copies of $\widetilde{e}_i^\varepsilon M$.*

In the case $\varepsilon_i(M) = 1$, this says $e_i M = \widetilde{e}_i M$ is irreducible.

The following lemma computes the action of e_i on an induced module. It is a special case of the "Mackey formula." We refer the reader to [**G**] or [**GV**] for a proof.

Lemma 2.2. *Let A be an irreducible H_a-module and B be an irreducible H_b-module. Let $n = a + b$. The following is an exact sequence:*

$$0 \to \mathrm{Ind}_{a-1,b}^{n-1} e_i A \boxtimes B \to e_i(\mathrm{Ind}_{a,b}^n A \boxtimes B) \to \mathrm{Ind}_{a,b-1}^{n-1} A \boxtimes e_i B \to 0$$

2.4. Propositions on induced modules.

Proposition 4. *Let A be an irreducible H_a-module and B be an irreducible H_b-module. Let $n = a + b$. Then $\mathrm{Ind}_{H_a \otimes H_b}^{H_n} A \boxtimes B \simeq \widehat{\mathrm{Ind}}_{H_b \otimes H_a}^{H_n} B \boxtimes A$, where $n = a + b$.*

Proposition 5. *Let $N = \mathrm{cosoc}\, \mathrm{Ind}_{a,b}^n A \boxtimes B$ where $n = a + b$. Then $N \simeq \mathrm{soc}\, \mathrm{Ind}_{b,a}^n B \boxtimes A \simeq \mathrm{soc}\, \widehat{\mathrm{Ind}}_{a,b}^n A \boxtimes B$.*

Lemma 2.3.(1) *If $\mathrm{Ind}_{a,b}^n A \boxtimes B$ is irreducible, then $\mathrm{Ind}_{a,b}^n A \boxtimes B \simeq \mathrm{Ind}_{b,a}^n B \boxtimes A$.*
(2) *If $\mathrm{Ind}_{a,b}^n A \boxtimes B$ has irreducible cosocle occurring with multiplicity one as a composition factor and $\mathrm{Ind}_{a,b}^n A \boxtimes B \simeq \mathrm{Ind}_{b,a}^n B \boxtimes A$, then $\mathrm{Ind}_{a,b}^n A \boxtimes B$ is irreducible.*

Proof. The first statement follows from results in [**GV**] and [**G**]. We remark that there is always a nonzero map $\mathrm{Ind}\, A \boxtimes B \to \mathrm{Ind}\, B \boxtimes A$. In fact, these modules always have the same composition factors (with multiplicity).

For the second statement, let $M = \mathrm{cosoc}\, \mathrm{Ind}\, A \boxtimes B$. Then also $M \simeq \mathrm{soc}\, \mathrm{Ind}\, B \boxtimes A \simeq \mathrm{soc}\, \mathrm{Ind}\, A \boxtimes B$ by proposition 5 and by hypothesis, so in particular M is a submodule. Since M occurs with multiplicity one, it cannot then be a quotient of $(\mathrm{Ind}\, A \boxtimes B)/M$. But because $M = \mathrm{cosoc}\, \mathrm{Ind}\, A \boxtimes B$, if $(\mathrm{Ind}\, A \boxtimes B)/M \neq 0$, it must be a quotient. Hence $\mathrm{Ind}\, A \boxtimes B = M$ is irreducible. □

3. Partitions

Recall that μ is a *partition* of n with length k if $\mu = (\mu_1, \ldots, \mu_k)$ such that $\sum_m \mu_m = n$ and $\mu_1 \geq \cdots \geq \mu_k > 0$. We write $|\mu| = n$.

Given an integer h, we can associate to μ a coloring of its Young diagram, which we will denote $\mu[h]$, as follows. The Young diagram associated to μ consists of μ_1 boxes in the first row, μ_2 boxes in the second, etc. Fill the (x, y) box with $h + y - x$.

h	$h+1$	$\cdots$		$\cdots$	b_1
$h-1$	h	$\cdots$	b_2		
$\vdots$					
$h-k+1$	$\cdots$	b_k			

Here, $b_m = h + \mu_m - m$. Notice the fillings are constant down the diagonals.

We will write μ for $\mu[h]$ when it is clear from context that μ is colored by h.

We say $\mu[h]$ has a *removable* i-box if you can remove an i-filled box $\boxed{\text{i}}$ from its h-colored Young diagram such that the result is again the diagram of a partition (colored by h). If so, let $\widetilde{\mathbf{e}}_{\mathbf{i}}\mu$ be that new partition of $n-1$. In the case that μ has a removable i-box, we say $\varepsilon_i(\mu) = 1$. Otherwise, $\varepsilon_i(\mu) = 0$ and $\widetilde{\mathbf{e}}_{\mathbf{i}}\mu = \mathbf{0}$.

We say $\mu[h]$ has an *addable* i-box if you can add an i-filled box to the diagram such that the result is again the diagram of a partition (colored by h). If μ has an addable i-box, we write $\varphi_i(\mu) = 1$. Otherwise, $\varphi_i(\mu) = 0$.

Observe that for all i, $\varepsilon_i(\mu) \leq 1$, $\varphi_i(\mu) \leq 1$ and they cannot both be nonzero, i.e. $\varepsilon_i(\mu) \cdot \varphi_i(\mu) = 0$.

Definition 1. Given an h-colored partition μ of n, define an irreducible H_n-module

$$N_\mu = N_{\mu[h]} = \operatorname{cosoc} \operatorname{Ind}^n_{\mu_1,\ldots,\mu_k}(q^h \cdots q^{b_1}) \boxtimes (q^{h-1} \cdots q^{b_2}) \boxtimes \cdots \boxtimes (q^{h-k+1} \cdots q^{b_k})$$

where $(q^\alpha q^{\alpha+1} q^{\alpha+2} \cdots q^\beta)$ denotes the 1-dimensional $H_{\beta-\alpha+1}$-module on which all $T_j - q$ and $X_j - q^{\alpha+j-1}$ vanish.

A proof of the following fact can be found in [**V**].

Theorem 3.1. *Let μ be a partition of n. Then $N_{\mu[h]}$ is irreducible and is an $H_n^{\Lambda_h}$-module.*

In other words, the $N_{\mu[h]}$ are the evaluation modules $S_\mu(q^h)$ of [**LNT**]. In the case $h = 0$, these are the Specht modules of $H_n^{\text{fin}} = H_n^{\Lambda_0}$.

4. Multipartitions

We will call an r-tuple of partitions $\underline{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$ a *multipartition* of n if $n = \sum_m |\mu^{(m)}|$.

Likewise, given a multipartition $\underline{\mu}$ and a weight $\lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_r}$ with $i_1 \geq i_2 \geq \cdots \geq i_r$, we will say $\underline{\mu}$ is colored by λ if $\mu^{(m)}$ is colored by i_m. Denote this as $\underline{\mu}[\lambda]$.

If $\underline{\mu}[\lambda]$ satisfies the condition

$$\mu^{(t)}_{i_t - i_{t+1} + x} \leq \mu^{(t+1)}_x \quad \text{for all } x \geq 1,\ 1 \leq t \leq r-1 \tag{4.1}$$

then we will say it is a *Kleshchev multipartition*.

In other words, if two partitions have a row beginning with the same filling, (even if it is empty) then the rightmost row must be the longer.

EXAMPLE 1. Let $\lambda = \Lambda_0 + 3\Lambda_3 + \Lambda_4$, $\underline{\mu} = ((3), \emptyset, (3,3), (3,3,1,1), (3,1))$. Then $\underline{\mu}[\lambda]$ is a Kleshchev multipartition.

$$\underline{\mu} = \left(\begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline \end{array}, \quad \emptyset, \quad \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 & 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1} 0 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 \\ \cline{1-1} \end{array} \right)$$

The only multipartitions we will consider in Theorem 5.2 will be Kleshchev multipartitions.

The reader may recognize that $\underline{\mu}[\lambda]$ satisfies (4.1) iff $\mu^{(r)} \otimes \cdots \otimes \mu^{(1)}$ is in the connected component of $\emptyset \otimes \cdots \otimes \emptyset$ in the tensor product of crystal graphs $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$. For a crystal-theoretic interpretation of the theorems below, see [**V**] or [**G**].

RULE (i-signature rule). Given a multipartition $\underline{\mu}$ and weight $\lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_r}$ with $i_1 \geq i_2 \geq \cdots \geq i_r$ we define $\widetilde{\mathbf{e}}_{\mathbf{i}}\underline{\mu} = \widetilde{\mathbf{e}}_{\mathbf{i}}(\underline{\mu}[\lambda])$ as follows. If $\mu^{(k)}[i_k]$ has a removable i box, write the symbol $-$. If $\mu^{(k)}[i_k]$ has an addable i box, write $+$. Otherwise, write a blank. This word of length r is the *i-signature* of $\underline{\mu}$.

In this word repeatedly cancel all occurrences of $-+$. In the remaining uncanceled symbols $++\cdots+-\cdots-$, we remove the i-box from the $\mu^{(k)}$ that corresponds to the leftmost $-$ symbol, if one exists. In other words, replace $\mu^{(k)}$ with $\widetilde{\mathbf{e}}_{\mathbf{i}}\mu^{(k)}$. The resulting colored multipartition is $\widetilde{\mathbf{e}}_{\mathbf{i}}\underline{\mu}$. Otherwise, $\widetilde{\mathbf{e}}_{\mathbf{i}}\underline{\mu} = \mathbf{0}$.

We note that $\widetilde{\mathbf{e}}_{\mathbf{i}}\underline{\mu}$ has the same i-signature as $\underline{\mu}$ except that the leftmost uncanceled $-$ has been changed to $+$.

In example 1 above where $\lambda = \Lambda_0 + 3\Lambda_3 + \Lambda_4$ and $\underline{\mu} = ((3), \emptyset, (3,3), (3,3,1,1), (3,1))$, then $\widetilde{\mathbf{e}}_{-\mathbf{1}}\underline{\mu} = ((3), \emptyset, (3,3), (3,3,1,1), (3))$ but $\widetilde{\mathbf{e}}_{\mathbf{0}}\underline{\mu} = \mathbf{0}$.

DEFINITION 2. Define $N_{\underline{\mu}} = N_{\underline{\mu}[\lambda]} = \operatorname{cosoc} \operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$.

By abuse of notation, we will often just write μ for $N_{\mu[h]}$, and so just

$$\operatorname{Ind} \underline{\mu} \quad \text{or} \quad \operatorname{Ind} \mu^{(1)} \boxtimes \cdots \boxtimes \mu^{(r)}$$

for $\operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$.

Here we have written $\operatorname{Ind}(\mu, \nu)$ or $\operatorname{Ind} \mu[h] \boxtimes \nu[k]$ for what [**LNT**] denotes $S_\mu(q^h) \odot S_\nu(q^k)$.

When $\mu = \emptyset$, $N_\emptyset = \mathbf{1}$, the trivial module of $H_0 = R$. This is not to be confused with the symbol $\mathbf{0}$ for which $N_{\mathbf{0}} = 0$.

We quote some results from [**V**].

THEOREM 4.1. *Fix λ. Let $\underline{\mu}[\lambda]$ be a Kleshchev multipartition, i.e. $\underline{\mu}$ satisfies* (4.1). *Then the modules*

$$N_{\underline{\mu}[\lambda]} = \operatorname{cosoc} \operatorname{Ind} \underline{\mu} = \operatorname{cosoc} \operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$$

are irreducible, distinct, and range over all irreducible H_n^λ-modules, $n \geq 0$. Consequently, Kleshchev multipartitions parameterize the irreducible H_n^λ-modules.

In the above, we referred to $N_{\underline{\mu}[\lambda]}$ as an H_n^λ-module, when really it is constructed as an H_n-module on which I_λ vanishes. We view it as an H_n-module in the rest of the paper.

THEOREM 4.2. *$N_{\underline{\mu}[\lambda]}$ occurs with multiplicity one as a composition factor of* Ind $\underline{\mu}$.

THEOREM 4.3. *Let $\underline{\mu}[\lambda]$ be a Kleshchev multipartition. Then*

$$\widetilde{e}_i N_{\underline{\mu}[\lambda]} = N_{\widetilde{\mathbf{e}}_\mathbf{i}(\underline{\mu}[\lambda])}.$$

If $\underline{\mu}[\lambda]$ is not a Kleshchev multipartition (does not satisfy (4.1)), then cosoc Ind $\underline{\mu}$ need not be irreducible. Furthermore, Theorem 4.3 need not hold for arbitrary colored multipartitions.

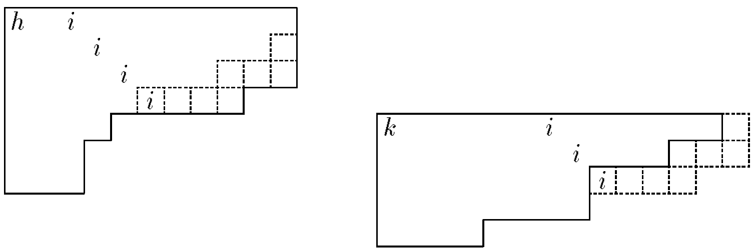
5. Main result

5.1. More definitions.

DEFINITION 3. We say $\mathbf{b}$ is a *border-strip* of length ℓ of the h-colored partition μ if it consists of boxes with consecutive fillings $\{i, i+1, \ldots, i+\ell-1\}$ that all lie on the border of μ. In other words, a box in $\mathbf{b}$ is always the farthest southeast on its diagonal of boxes with the same filling, and so has no box diagonally southeast of it.

DEFINITION 4. Given two colored partitions $\mu[h]$ and $\nu[k]$, we say a border strip $\mathbf{b}$ of μ is *movable* from μ to ν if: first, $\mathbf{b}$ is a border strip of μ such that its removal results again in a partition colored by h, and second, that $\mathbf{b}$ can be added to ν resulting in a partition still colored by k.

Note that when we add $\mathbf{b}$ to ν, we require that it retains its shape as well as its fillings.



Also note that not every border-strip of μ is removable. If $\mathbf{b}$ is removable from μ it is also sometimes referred to as an ℓ-rim hook in the literature.

In example 1, $\boxed{3}\boxed{4}$ is movable from $\mu^{(4)}$ to $\mu^{(5)}$. However, although $\boxed{4}\boxed{5}\boxed{6}$ is removable from $\mu^{(1)}$, it is not movable from $\mu^{(1)}$ to $\mu^{(2)}$.

5.2. Main theorems.

LEMMA 5.1. *Let $\underline{\mu}[\lambda]$ be a Kleshchev multipartition.* Ind $\underline{\mu}$ *is irreducible* $\iff$ Ind $\underline{\mu} \simeq$ Ind $\mu^{(\sigma_1)} \boxtimes \cdots \boxtimes \mu^{(\sigma_r)}$ *for all* $\sigma \in S_r$.

PROOF. The forward direction follows from the first statement of lemma 2.3 by the transitivity and exactness of induction, and since S_r is generated by adjacent transpositions.

For the converse, induct on r. The case $r = 2$ is given by the second statement of lemma 2.3, since Ind $\underline{\mu}$ satisfies its hypotheses by theorem 4.2. For $r > 2$, again use the lemma, letting $A = N_{\mu^{(1)}}$ and $B = N_{(\mu^{(2)}, \ldots, \mu^{(r)})}$ and observing that $(\mu^{(2)}, \ldots, \mu^{(r)})$ is also a Kleshchev multipartition. $\square$

THEOREM 5.2 (Main Theorem). *Let $\underline{\mu}[\lambda] = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)})$ be a Kleshchev multipartition of n, colored by λ. The H_n^λ-module $N_{\underline{\mu}[\lambda]}$ is reducible $\iff$ for some $i < j$ there is a movable border strip from $\mu^{(i)}$ to $\mu^{(j)}$.*

PROOF. By lemma 5.1, it suffices to prove the theorem for the case $r = 2$ since $(\mu^{(i)}, \mu^{(j)})$ is again a Kleshchev multipartition.

If $\mathrm{Ind}\, \mu^{(i)} \boxtimes \mu^{(j)}$ is reducible, then so is $\mathrm{Ind}\, \underline{\mu}$. If all $\mathrm{Ind}\, \mu^{(i)} \boxtimes \mu^{(j)}$ are irreducible then we can permute all i and j and so by lemma 5.1 $\mathrm{Ind}\, \underline{\mu}$ is irreducible.

The case $r = 2$ is given in theorem 5.4 below. □

LEMMA 5.3. *Let (μ, ν) be a Kleshchev multipartition (colored by λ) such that $\varepsilon_i(\mu) = \varphi_i(\nu) = 1$. Then $\mathrm{Ind}\, \mu \boxtimes \nu$ is reducible.*

PROOF. Lemma 2.2 shows that $0 \neq \mathrm{Ind}\, e_i \mu \boxtimes \nu \subseteq e_i(\mathrm{Ind}\, \mu \boxtimes \nu)$. If $\mathrm{Ind}\, \mu \boxtimes \nu$ were irreducible, then $\mathrm{Ind}\, \mu \boxtimes \nu = \mathrm{cosoc}\, \mathrm{Ind}\, \mu \boxtimes \nu = N_{\mu\nu}$. By the i-signature rule 4, (μ, ν) has i-signature $-+$ and so $\varepsilon_i(N_{\mu\nu}) = 0$, showing $e_i N_{\mu\nu} = 0$. Contradiction. □

THEOREM 5.4. *Let (μ, ν) be a Kleshchev multipartition of n, colored by $\lambda = \Lambda_{i_1} + \Lambda_{i_2}$. The H_n^λ-module $\mathrm{Ind}\, \mu \boxtimes \nu$ is reducible $\iff$ there is a movable border strip from μ to ν.*

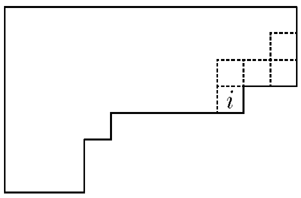
PROOF. Write $N = \mathrm{cosoc}\, \mathrm{Ind}\, \mu \boxtimes \nu$. So $\mathrm{Ind}\, \mu \boxtimes \nu = N \iff \mathrm{Ind}\, \mu \boxtimes \nu$ is irreducible.

$\Longleftarrow$: Suppose $\mathbf{b}$ is a movable border strip from μ to ν. We will prove $\mathrm{Ind}\, \mu \boxtimes \nu = \mathrm{Ind}\, N_{\mu[i_1]} \boxtimes N_{\nu[i_2]}$ is reducible. We will induct on the length ℓ of $\mathbf{b}$.

When $\ell = 1$ this means μ has a removable $\boxed{\mathrm{i}}$-box and that ν has an addable $\boxed{\mathrm{i}}$-box. By lemma 5.3, $\mathrm{Ind}\, \mu \boxtimes \nu$ is reducible.

Now assume the statement holds for all movable border strips of length $< \ell$. Let $\{i, i+1, \ldots, i+\ell-1\}$ be the fillings of the boxes of $\mathbf{b}$. Assume for the sake of contradiction that $N = \mathrm{Ind}\, \mu \boxtimes \nu$. There are two cases.

Case 1: $i + 1$ is above i in $\mathbf{b}$.

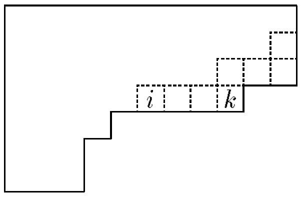


Then in fact we can remove that $\boxed{\mathrm{i}}$, i.e. $\varepsilon_i(\mu) = 1$. Also $\varepsilon_i(\nu) = 0$ since to add $\mathbf{b}$ to ν there must be an $i - 1$ beneath the i on the border of ν. This also shows $\varphi_i(\nu) = 0$ (although if it were nonzero, we could apply lemma 5.3). Hence $\varepsilon_i(N) = 1$ and so $\widetilde{e}_i N = e_i N$ is irreducible and equal to $e_i(\mathrm{Ind}\, \mu \boxtimes \nu) = \mathrm{Ind}\, \widetilde{\mathbf{e}}_{\mathbf{i}} \mu \boxtimes \nu$. The multipartition $(\widetilde{\mathbf{e}}_{\mathbf{i}} \mu, \nu)$ is again a Kleshchev multipartition.

Now $\mathbf{b} \backslash \boxed{\mathrm{i}}$ is a movable border strip of length $\ell - 1$ from $\widetilde{\mathbf{e}}_{\mathbf{i}} \mu$ to ν. By induction, $\mathrm{Ind}\, \widetilde{\mathbf{e}}_{\mathbf{i}} \mu \boxtimes \nu$ is reducible. Contradiction.

Case 2: $i + 1$ is to the right of i in $\mathbf{b}$.

Let $\{i, \ldots, k\}$ be the fillings in the bottom row of $\mathbf{b}$.

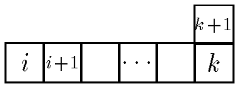


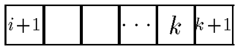
If $k = i+\ell-1$, then $\mathbf{b}$ is contained in a single row. Since $\mathbf{b}$ is movable, $\varepsilon_k(\nu) = \varphi_k(\nu) = 0$ and $\varepsilon_k(\mu) = 1$. As above, $e_k(\operatorname{Ind} \mu \boxtimes \nu) = \operatorname{Ind} \widetilde{\mathbf{e}}_{\mathbf{k}}\mu \boxtimes \nu = e_k N = \widetilde{e}_k N$ is irreducible. However, $\mathbf{b} \setminus \boxed{\text{k}}$ is a movable border strip of length $\ell - 1$ from $\widetilde{\mathbf{e}}_{\mathbf{k}}\mu$ to ν, so that by the inductive hypothesis, $\operatorname{Ind} \widetilde{\mathbf{e}}_{\mathbf{k}}\mu \boxtimes \nu$ is reducible. Contradiction.

If $k < i + \ell - 1$, then $k + 1$ sits above k in $\mathbf{b}$, and $\varepsilon_k(\nu) = \varepsilon_k(\mu) = 1$.

After removing $\boxed{\text{k}}$ from both shapes, one can continue removing $k - 1$ and so on, down to $i + 1$. Now one can remove $\boxed{\text{i}}$ only from the first shape. In other words, $\varepsilon_i(\widetilde{\mathbf{e}}_{\mathbf{i+1}} \cdots \widetilde{\mathbf{e}}_{\mathbf{k}}\mu) = 1$ and $\varepsilon_i(\widetilde{\mathbf{e}}_{\mathbf{i+1}} \cdots \widetilde{\mathbf{e}}_{\mathbf{k}}\nu) = 0$. Also since we have just removed an $i + 1$, $\varphi_i(\widetilde{\mathbf{e}}_{\mathbf{i+1}} \cdots \widetilde{\mathbf{e}}_{\mathbf{k}}\nu) = 0$. Let $\mu' = \widetilde{\mathbf{e}}_{\mathbf{i}}\widetilde{\mathbf{e}}_{\mathbf{i+1}} \cdots \widetilde{\mathbf{e}}_{\mathbf{k}}\mu$ and $\nu' = \widetilde{\mathbf{e}}_{\mathbf{i+1}} \cdots \widetilde{\mathbf{e}}_{\mathbf{k}}\nu$. Keeping track of all i-signatures and using proposition 3, we get that $\operatorname{Ind} \mu' \boxtimes \nu' \simeq \widetilde{e}_i \widetilde{e}_{i+1}^2 \cdots \widetilde{e}_k^2 N$ is irreducible.

Consider the border strip $\mathbf{b}'$ of μ' with filling $\{i+1, \ldots, i+\ell-1\}$. It differs from $\mathbf{b}$ by replacing



with


.

By the inductive hypothesis, $\operatorname{Ind} \mu' \boxtimes \nu'$ is reducible. Contradiction.

(Notice that in the case that μ is itself a movable border strip and $\nu = \emptyset$, we must have $i_1 = i_2$ and so (μ, ν) is not a Kleshchev multipartition.)

$\Longrightarrow$: Now we will show that if $\operatorname{Ind} \mu \boxtimes \nu$ is reducible, a movable border strip exists. This time, induct on n. The base case is given by the computation in [**GV**] that $\operatorname{Ind} \boxed{\text{i}} \boxtimes \boxed{\text{j}} = \operatorname{Ind} q^i \boxtimes q^j$ is reducible if and only if $i = j \pm 1$. Then let the border strip $\mathbf{b} = \boxed{\text{i}}$.

If $\operatorname{Ind} \mu \boxtimes \nu$ is reducible, then we have an exact sequence

$$0 \to K \to \operatorname{Ind} \mu \boxtimes \nu \to N \to 0$$

with $K \neq 0$. Thus, there is some i such that $e_i(K) \neq 0$.

There are three cases.

Case 1: $\varepsilon_i(\mu) = \varepsilon_i(\nu) = 1$.

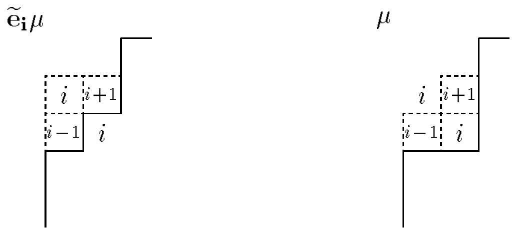
First we'll show $e_i^2 K \neq 0$. Let $L \subseteq K$ be simple. Since $\varepsilon_i(\mu) = 1$ then $\varphi_i(\mu) = 0$, so that $\operatorname{Ind} \boxed{\text{i}} \boxtimes \mu \simeq \operatorname{Ind} \mu \boxtimes \boxed{\text{i}}$ by a result in [**G**]. Since $L \subseteq \operatorname{soc} \operatorname{Ind} \mu \boxtimes \nu \simeq \operatorname{cosoc} \operatorname{Ind} \nu \boxtimes \mu$, then $\varepsilon_i(L) = 2$. Thus

$$0 \to K' \to \operatorname{Ind} \widetilde{\mathbf{e}}_{\mathbf{i}}\mu \boxtimes \widetilde{\mathbf{e}}_{\mathbf{i}}\nu \to \widetilde{e}_i^2 N \to 0$$

is exact, where $0 \neq K' \subseteq e_i^2 K$. Then $\operatorname{Ind} \widetilde{\mathbf{e}}_{\mathbf{i}}\mu \boxtimes \widetilde{\mathbf{e}}_{\mathbf{i}}\nu$ is reducible. By the inductive hypothesis, there is a border strip $\mathbf{b}$ movable from $\widetilde{\mathbf{e}}_{\mathbf{i}}\mu$ to $\widetilde{\mathbf{e}}_{\mathbf{i}}\nu$.

Suppose $\mathbf{b}$ does not contain any boxes filled with $i, i \pm 1$. Then the same $\mathbf{b}$ can be moved from μ to ν.

Suppose $\mathbf{b}$ contains an $\boxed{\mathrm{i}}$. Then the i removed from μ and ν had to have an $i+1$ above it and an $i-1$ to its left, which then surround the i in $\mathbf{b}$. So for $\mathbf{b}$ to be removable from $\widetilde{\mathbf{e}}_{\mathbf{i}}\mu$, it must also contain $i-1, i, i+1$.



Let $\mathbf{b}'$ be the border strip of μ which differs from $\mathbf{b}$ only in that the i it contains is instead the i we had removed from μ. Then $\mathbf{b}'$ is movable from μ to ν.

The remaining case is that $\mathbf{b}$ does not contain i and contains exactly one of $i \pm 1$. Since its fillings are consecutive, it cannot contain both of $i \pm 1$ without also containing i. Then let $\mathbf{b}'$ be the border strip of μ which differs from $\mathbf{b}$ only in that it does contain the i we removed from μ. Then $\mathbf{b}'$ is movable from μ to ν.

Case 2: $\varepsilon_i(\mu) = 1$, $\varepsilon_i(\nu) = 0$.
If $\varphi_i(\nu) = 1$, then let $\mathbf{b} = \boxed{\mathrm{i}}$ and we are done.

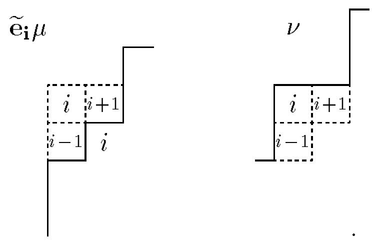
Otherwise $\varepsilon_i(N) = 1$ and so

$$\begin{array}{ccccccccc} 0 & \to & e_i K & \to & e_i(\mathrm{Ind}\,\mu \boxtimes \nu) & \to & e_i N & \to & 0 \\ & & & & \| & & \| & & \\ & & & & \mathrm{Ind}\,\widetilde{\mathbf{e}}_{\mathbf{i}}\mu \boxtimes \nu & & \widetilde{e}_i N & & \end{array}$$

is exact. We chose i so $e_i K \neq 0$, hence $\mathrm{Ind}\,\widetilde{\mathbf{e}}_{\mathbf{i}}\mu \boxtimes \nu$ is reducible. By the inductive hypothesis, there is a border strip $\mathbf{b}$ movable from $\widetilde{\mathbf{e}}_{\mathbf{i}}\mu$ to ν.

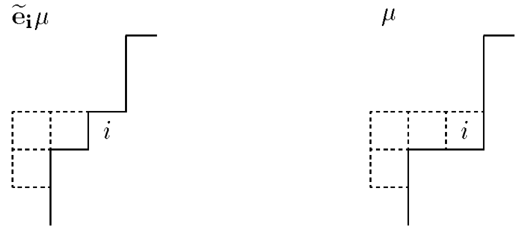
Suppose $\mathbf{b}$ contains none of $i-1, i, i+1$. Then the same $\mathbf{b}$ is movable from μ to ν.

Suppose $\mathbf{b}$ contains an $\boxed{\mathrm{i}}$. Then, as above, for $\mathbf{b}$ to be removable from $\widetilde{\mathbf{e}}_{\mathbf{i}}\mu$, it must also contain $i-1, i, i+1$



But if this $\mathbf{b}$ can be added to ν, it implies we can add an $\boxed{\mathrm{i}}$ to ν, i.e. $\varphi_i(\nu) = 1$, and we are done.

The remaining case is that $\mathbf{b}$ does not contain i, and contains exactly one of $i \pm 1$.



Let $\mathbf{b}'$ be the border strip of μ which differs from $\mathbf{b}$ only in that it also contains the i we removed from μ. Then $\mathbf{b}'$ is movable from μ to ν.

If the i on the border of ν did not have an $i-1$ below it nor an $i+1$ to its right, then it would be removable, contradicting $\varepsilon_i(\nu) = 0$. However, if they were both there, then $\varphi_i(\nu) = 1$. So it must be we have $i \pm 1$, but not $i \mp 1$ in those spots. For $\mathbf{b}$ to be addable to ν, it must contain $i \mp 1$. Since $i \pm 1$ is east or south of i, in fact this shows $\mathbf{b}'$ is addable to ν. The border strip $\mathbf{b}'$ is certainly removable from μ.

Case 3: $\varepsilon_i(\mu) = 0$, $\varepsilon_i(\nu) = 1$.
Just as in Case 2, $e_i(\mathrm{Ind}\, \mu \boxtimes \nu) = \mathrm{Ind}\, \mu \boxtimes \widetilde{\mathbf{e}}_{\mathbf{i}}\nu$ is reducible. By the inductive hypothesis, there is a border strip $\mathbf{b}$ movable from μ to $\widetilde{\mathbf{e}}_{\mathbf{i}}\nu$.

Suppose $\mathbf{b}$ does not contain any of $i-1, i, i+1$. Then $\mathbf{b}$ is movable from μ to ν.

If $\mathbf{b}$ contains $i \pm 1$, then it must also contain i since it is addable to $\widetilde{\mathbf{e}}_{\mathbf{i}}\nu$. And $\mathbf{b}$ cannot contain just i since $\varepsilon_i(\mu) = 0$. So in fact, $\mathbf{b}$ must contain at least two of $i-1, i, i+1$. Further, since $\varepsilon_i(\mu) = 0$, one of these must be south or east of i. Say $\mathbf{b}$ contains i and $i \pm 1$, with $i \pm 1$ east or south.

Then let $\mathbf{b}'$ be $\mathbf{b}$ truncated so it still contains $i \pm 1$, but not i. In other words, its filling are either $\{\ldots, i-2, i-1\}$ or $\{i+1, i+2, \ldots\}$. Because the $i \pm 1$ is east or south, $\mathbf{b}'$ is still removable from μ. And that $i \pm 1$ will land east or south of the removable i in ν, and so $\mathbf{b}'$ is addable to ν. Thus $\mathbf{b}'$ is movable from μ to ν. $\square$

We conclude with the following remark. When $\mathrm{char} R = 0$, the group algebra of $(\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$ is semisimple. By a further abuse of notation, its irreducible modules all have the form

$$\mathrm{Ind}\, \underline{\mu}$$

if we replace q^{i_k} with ζ^k where $\zeta \in R$ is a primitive r^{th} root of unity.

When $\lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_r}$ is such that the i_k are far apart compared to $n = \sum |\mu^{(i)}|$, then H_n^λ is semisimple, and we expect its representation theory to coincide with that of $(\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$. Indeed, if $\underline{\mu}$ is any r-tuple of partitions colored by λ, with $n = \sum |\mu^{(i)}|$, then $\mathrm{Ind}\, \underline{\mu}$ is irreducible. Any such $\underline{\mu}$ is a Kleshchev multipartition and satisfies the condition in theorem 5.2.

References

[A] Ariki, S. On the decomposition numbers of the Hecke algebra of $G(m,1,n)$. *J. Math. Kyoto Univ.* **36** (1996), no. 4, 789–808.

[A2] Ariki, S. On the classification of simple modules for cyclotomic Hecke algebras of $G(m,1,n)$ and Kleshchev multipartitions. preprint math.QA/9908004.

[AK] Ariki, S.; Koike, K. A Hecke algebra of $(Z/rZ) \wr S_n$ and construction of its irreducible representations. *Adv. Math.* **106** (1994), no. 2, 216–243.

[AM] Ariki, S.; Mathas, A. The number of simple modules of the Hecke algebras of type $G(r, 1, n)$. *Math. Z.* **233** (2000), no. 3, 601–623.

[B] Bernstein, I. N. Lectures on Smooth Representations of p-adic Groups. Harvard lecture notes 1991.

[BZ] Bernstein, I. N.; Zelevinsky, A. V. Induced representations of reductive p-adic groups. *I. Ann. Sci. École Norm. Sup.* **10** (1977), no. 4, 441–472.

[Br] Brundan, J. Modular branching rules and the Mullineux map for Hecke algebras of type A. *Proc. Lond. Math. Soc.* **77**(1998), 551-581.

[D] Drinfel'd, V. G. Degenerate affine Hecke algebras and Yangians. (Russian) *Funktsional. Anal. i Prilozhen.* **20** (1986), no. 1, 69–70.

[G] Grojnowski, I. Affine $\widehat{\mathfrak{s}l}_p$ controls the modular representation theory of the symmetric group and related Hecke algebras. Preprint. RT/9907129.

[G2] Grojnowski, I. Representations of affine Hecke algebras (and affine quantum GL_n) at roots of unity. *Internat. Math. Res. Notices* (1994), no. 5, 215-217.

[GV] Grojnowski, I.; Vazirani, M. Strong multiplicity one theorem for affine Hecke algebras of type A. *Journal of transformation groups.* **6** (2001), no. 2, 143-155.

[JMMO] Jimbo, M.; Misra, K.; Miwa, T.; Okado, M. Combinatorics of representations of $U_q(\widehat{\mathfrak{s}l}(n))$ at $q = 0$. *Comm. Math. Phys.* **136** (1991), no. 3, 543–566.

[K] Kashiwara, M. On crystal bases. *Representations of groups (Banff, AB, 1994)*, 155–197, CMS Conf. Proc., **16**, Amer. Math. Soc., Providence, RI, 1995.

[Ka] Kato, S. Irreducibility of principal series representations for Hecke algebras of affine type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), no. 3, 929–943 (1982).

[KL] Kazhdan, D.; Lusztig, G. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.* **87** (1987), no. 1, 153–215.

[Kv1] Kleshchev, A. Branching rules for modular representations of symmetric groups. II. *J. Reine Angew. Math.* **459** (1995), 163–212.

[Kv2] Kleshchev, A. Branching rules for modular representations of symmetric groups, III: some corollaries and a problem of Mullineux, *J. Lond. Math. Soc.* **54**(1996), 25-38.

[Kv3] Kleshchev, A. On decomposition numbers and branching coefficients for symmetric and special linear groups, *Proc. Lond. Math. Soc.* **75**(1997), 497-558.

[LLT] Lascoux, A.; Leclerc, B.; Thibon, J. Hecke algebras at roots of unity and crystal bases of quantum affine algebras. *Comm. Math. Phys.* **181** (1996), no. 1, 205–263.

[LNT] Leclerc, B.; Nazarov, M.; Thibon, J. Induced representations of affine Hecke algebras and canonical bases of quantum groups. QA/0011074.

[L] Lusztig, G. Affine Hecke algebras and their graded version. *J. Amer. Math. Soc.* **2** (1989), no. 3, 599–635.

[M] Mathas, A. Simple modules of Ariki-Koike algebras. *Proc. Pure Symp. Math.*,**63** (1998), 383-396.

[NT] Nazarov, M.; Tarasov, V. On irreducibility of tensor products of Yangian modules associated with skew Young diagrams. QA/0012039v2.

[V] Vazirani, M. Parameterizing Affine Hecke Algebra Modules: Bernstein-Zelevinsky Multisegments, Kleshchev Multipartitions, and Crystal Graphs. To appear in *Journal of transformation groups.*

[V2] Vazirani, M. *Irreducible Modules over the Affine Hecke Algebra: a Strong Multiplicity One Result.* PhD thesis. UC Berkeley, 1999.

[Z] Zelevinsky, A. V. Induced representations of reductive p-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup.* **13** (1980), no. 2, 165–210.

UC Berkeley, Department of Mathematics, 970 Evans Hall # 3840, Berkeley, CA 94720-3840

E-mail address: vazirani@math.berkeley.edu