Filtrations on the Mackey Decomposition for Cyclotomic Hecke Algebras

M. Vazirani
Department of Mathematics
Berkeley, CA 94720-3840
E-mail: vazirani@math.berkeley.edu

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In [G], Grojnowski defines functors $e_i : \operatorname{Rep} H_n^\lambda \to \operatorname{Rep} H_{n-1}^\lambda$ and $f_i : \operatorname{Rep} H_n^\lambda \to \operatorname{Rep} H_{n+1}^\lambda$ and shows that in the Grothendieck group $K(\operatorname{Rep} H_n^\lambda)$

$$[e_i, f_i]M = h_i(M)M,$$

where $h_i(M)$ is an integer depending only on λ and the central character of M. In this paper, we show that the above relation makes sense when M is an irreducible H_n^λ-module before passing to the Grothendieck group. That is, in the case $h_i(M) \geq 0$

$$e_i f_i M \simeq f_i e_i M \bigoplus h_i(M) M$$

and in the case $h_i(M) \leq 0$

$$e_i f_i M \bigoplus |h_i(M)|\, M \simeq f_i e_i M,$$

where for $k \geq 0$, kM denotes the direct sum $\bigoplus_{j=1}^{k} M$.

1. INTRODUCTION

Let H_n^λ (or $H_n^\lambda(q)$) denote the cyclotomic Hecke algebra, where λ is an r-tuple of nonzero elements from a field R, and $q \in R^\times$ as well. H_n^λ is a deformation of the group algebra of the wreath product $(\mathbb{Z}/r\mathbb{Z}) \wr S_n$ (and has dimension $r^n n!$). A precise definition is given in section 2 below. Certain λ can be identified with level r dominant integral weights of the affine Lie algebra $\widehat{\mathfrak{sl}}_\ell$ when $q^\ell = 1$ (or of $\mathfrak{gl}_\infty$ when the parameter q is not a root of unity). In this case, results of Ariki [A] and Grojnowski [G] show that the direct sum of Grothendieck groups

$$\bigoplus_n K(\operatorname{Rep} H_n^\lambda)$$

has the structure of an irreducible $\widehat{\mathfrak{sl}}_\ell$-module ($\mathfrak{gl}_\infty$-module) with highest weight λ. In particular, they each define operators e_i, f_i, and h_i and prove they satisfy the Chevalley relations. Grojnowski's operators come from functors $e_i : \operatorname{Rep} H_n^\lambda \to \operatorname{Rep} H_{n-1}^\lambda$ and $f_i : \operatorname{Rep} H_n^\lambda \to \operatorname{Rep} H_{n+1}^\lambda$, and many of his results come from a careful study of the structure of H_n^λ-modules *before* passing to the Grothendieck group $K(\operatorname{Rep} H_n^\lambda)$. However, one must pass to the Grothendieck group to show the e_i and f_i satisfy the Serre relations, and to define the h_i.

Here, we show that an analogue of

$$[e_i, f_i] = h_i$$

holds in $\operatorname{Rep} H_n^\lambda$, before passing to the Grothendieck group.

Given an irreducible module M, we prove that one of the modules $e_i f_i M$ and $f_i e_i M$ is a direct summand of the other and that the quotient is a direct sum of copies of M.

The multiplicity of M in this quotient is $|\varphi_i(M) - \varepsilon_i(M)|$. The quantities $\varphi_i(M)$ and $\varepsilon_i(M)$ are defined in terms of the functors f_i and e_i, and have many equivalent characterizations as outlined in Theorems 2.3 and 2.4 from [G] cited below.

The map that gives the decomposition

$$\begin{aligned} & e_i f_i M \simeq f_i e_i M \bigoplus |\varphi_i(M) - \varepsilon_i(M)|\, M \\ \text{or} \qquad & e_i f_i M \bigoplus |\varphi_i(M) - \varepsilon_i(M)|\, M \simeq f_i e_i M \end{aligned}$$

is a refinement of the cyclotomic Mackey decomposition,

$$rM \bigoplus \operatorname{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} \operatorname{Res}_{H_{n-1}^\lambda}^{H_n^\lambda} M \simeq \operatorname{Res}_{H_n^\lambda}^{H_{n+1}^\lambda} \operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} M.$$

The main new ingredient is the study of canonical filtrations on the modules $e_i M, f_i M, e_i f_i M$, and $f_i e_i M$ that are induced by the action of certain central elements. This enables us to refine the proofs of [G] sections 9 and 12 and obtain Theorem 7.1. In particular, we describe how copies of the irreducible modules $\widetilde{e}_i M$, $\widetilde{f}_i M$, and M sit in these filtrations and show the center acts on them as nilpotent Jordan blocks of sizes $\varepsilon_i(M)$ and $\varphi_i(M)$ and in terms of the tensor product of two such blocks.

We warn the reader of a minor shift in notation: that we will consider functors e_a and f_a where $a \in R^\times$, generalizing the case $a = q^i$ which are denoted e_i and f_i above (and in [G]).

2. DEFINITIONS

The definitions and propositions below are repeated from [G] or [GV], and we refer the reader there for proofs and for placing them in a larger context.

Throughout the paper we fix an algebraically closed field R, and an invertible element $q \in R^\times$. We further assume that $q \neq 1$. For the case $q = 1$ we refer the reader to the Appendix of [G].

If A is an R-algebra, we write $\operatorname{Rep} A$ for the category of left A-modules which are *finite dimensional* as R-modules. We also write $Z(A)$ for the center of the algebra A.

Recall that the socle of a module M, denoted $\operatorname{soc}(M)$, is the largest semisimple submodule of M, and that the cosocle of M, denoted $\operatorname{cosoc}(M)$, is its largest semisimple quotient. If M is an A-module and M' is an A'-module, $M \boxtimes M'$ denotes the $A \otimes A'$-module which is isomorphic to $M \otimes_R M'$ as an R-module and has $A \otimes A'$ action given by

$$(a \otimes a') \cdot (m \otimes m') = am \otimes a'm'.$$

For algebras $A \subset B$, we let Res denote restriction, and Ind denote induction. We take Ind to be the left adjoint of restriction, i.e.

$$\operatorname{Ind}_A^B M = B \otimes_A M,$$

whereas the right adjoint is given by

$$\check{\operatorname{Ind}}_A^B M = \operatorname{Hom}_A(B, M).$$

2.1. H_n^{fin}, H_n and H_n^λ

The *finite Hecke algebra*, H_n^{fin} is the R-algebra with generators

$$T_1, \ldots, T_{n-1}$$

and relations

braid relations	$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \ \lvert i-j\rvert > 1$
quadratic relations	$(T_i + 1)(T_i - q) = 0.$

The *affine Hecke algebra* H_n (or H_n^{aff}) is the R-algebra, which as an R-module is isomorphic to

$$H_n^{\mathrm{fin}} \otimes_R R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}].$$

The algebra structure is given by requiring that H_n^{fin} and $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ are subalgebras, and that

$$T_i X_i T_i = q X_{i+1}, \qquad X_i T_j = T_j X_i \text{ if } i \neq j, j+1$$

Proposition 2.1. *(Bernstein) The center of H_n, $Z(H_n)$, is isomorphic to symmetric Laurent polynomials.*

$$Z(H_n) \simeq R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}$$

From this one can deduce that irreducible H_n-modules are finite dimensional.

It also follows that if M is an irreducible H_n-module, its central character can be identified with an orbit $s = S_n \cdot (s_1, \ldots, s_n)$ in $(R^\times)^n$.

In this case, there is a simultaneous $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ eigenvector $v \in M$ and $\sigma \in S_n$ such that $X_i v = s_{\sigma(i)} v$ for all i. (σ and v are not unique.) Observe that any indecomposable module has a well-defined central character.

The *cyclotomic Hecke algebra* or *Ariki-Koike algebra* [AK] H_n^λ is the quotient H_n / I_λ, where I_λ is the ideal generated by $\prod_{j=1}^r (X_1 - \lambda_j)$, and $\lambda = (\lambda_1, \ldots, \lambda_r)$ is an r-tuple in $(R^\times)^r$.

In other words, H_n^λ-modules are those H_n-modules on which X_1 has eigenvalues in $\{\lambda_j\}$ with Jordan block size bounded by the multiplicities in λ.

The case that all $\lambda_j \in \{q^i \mid i \in \mathbb{Z}\}$ is of special interest. In particular, one can then associate λ with a weight of $\widehat{\mathfrak{sl}}_\ell$ if q is a primitive ℓ^{th} root of unity, or a weight of $\mathfrak{gl}_\infty$ otherwise. (And so the result $\bigoplus_n K(\operatorname{Rep}_q H_n) \simeq L(\lambda)$ from [A],[G] mentioned in the introduction holds).

The surjective algebra homomorphism

$$\operatorname{ev}_\lambda : H_n \to H_n / I_\lambda$$

induces an exact functor $\operatorname{ev}_\lambda^* : \operatorname{Rep} H_n^\lambda \to \operatorname{Rep} H_n$, with left adjoint $\operatorname{pr}_\lambda$, and right adjoint $\check{\operatorname{pr}}_\lambda$. The functor $\operatorname{pr}_\lambda$ picks off the largest quotient of N that is an H_n^λ-module, and can be defined via

$$\operatorname{pr}_\lambda(N) = N / I_\lambda N.$$

The functor $\check{\operatorname{pr}}_\lambda$ picks off the largest submodule of N that is an H_n^λ-module, i.e. that is killed by I_λ, and so can be defined via $\check{\operatorname{pr}}_\lambda(N) = N^{I_\lambda}$.

Given an H_n^λ-module M, we will often find it more useful to view it as an H_n-module. That is, we will confuse it with $\operatorname{ev}_\lambda^* M$, but be sloppy about writing the $\operatorname{ev}_\lambda^*$.

Observe that there is an embedding $H_{n_1} \otimes H_{n_2} \otimes \cdots \otimes H_{n_k} \hookrightarrow H_n$ if the n_j are positive integers summing to n. When we induce and restrict modules between these algebras, we will often abbreviate as $\operatorname{Ind}_{n_1, \ldots, n_k}^n$ and $\operatorname{Res}_{n_1, \ldots, n_k}^n$ or simply as Ind and Res.

We will be more careful with subscripts and superscripts on the rare occasion when inducing in $\operatorname{Rep} H_n^\lambda$. In general, we avoid $\operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda}$, as the power of the definitions of e_a and f_a below come from working within the affine Hecke algebra.

2.2. The functor e_a

Let N be any module in $\operatorname{Rep} H_n$. Define $\Delta_a N$ to be the generalized a-eigenspace of X_n in $\operatorname{Res}_{H_{n-1} \otimes H_1}^{H_n} N$. Then $\Delta_a N$ is an $H_{n-1} \otimes H_1$-module because $X_n - a \subseteq Z(H_{n-1} \otimes H_1)$. Furthermore

$$\mathrm{Res}^{H_n}_{H_{n-1}\otimes H_1} N = \bigoplus_{a\in R^\times} \Delta_a N.$$

Define a functor $e_a : \mathrm{Rep}\, H_n \to \mathrm{Rep}\, H_{n-1}$ by

$$e_a N = \mathrm{Res}^{H_{n-1}\otimes H_1}_{H_{n-1}} (\Delta_a N).$$

We list without proof various properties of e_a below.

We remind the reader that e_a is an exact functor $\mathrm{Rep}\, H_n \to \mathrm{Rep}\, H_{n-1}$ (although we can also view it as a functor $\mathrm{Rep}\, H_n^\lambda \to \mathrm{Rep}\, H_{n-1}^\lambda$ because $\mathrm{ev}_\lambda^* e_a = e_a \mathrm{ev}_\lambda^*$) and

$$\mathrm{Res}^{H_n}_{H_{n-1}} N = \bigoplus_{a\in R^\times} e_a N.$$

If N has central character $s = S_n \cdot (s_1, \ldots, s_n)$ and $e_a N \neq 0$ then $e_a N$ has central character $s - a = S_{n-1} \cdot (s_1, \ldots, s_{n-1})$, making the assumption $s_n = a$. In this case, $e_a N$ is *the* direct summand of $\mathrm{Res}^n_{n-1,1} N$ with central character $s - a$.

Now, let M be an *irreducible* module in $\mathrm{Rep}\, H_n$. Define

$$\widetilde{e}_a M = \mathrm{soc}\, e_a M.$$

It is the main theorem of [GV] that

Theorem 2.2. *Let M be an irreducible H_n-module. Then $\widetilde{e}_a M$ is either irreducible or zero. Further, $\widetilde{e}_a M \simeq \mathrm{cosoc}\, e_a M$.*

Recall that if M and N are irreducible modules such that $\widetilde{e}_a M \simeq \widetilde{e}_a N \neq 0$ then $M \simeq N$.

Define

$$\varepsilon_a(M) = \max\{m \mid e_a^m M \neq 0\} = \max\{m \mid \widetilde{e}_a^m M \neq 0\}.$$

We remark that $\widetilde{e}_a M = 0 \iff e_a M = 0 \iff \varepsilon_a(M) = 0$.

We recall equivalent characterizations of $\varepsilon_a(M)$ from Theorem 9.13 of [G].

Theorem 2.3. *Let M be an irreducible H_n-module. Then*

(i) $\varepsilon_a(M)$ is the maximal size of a nilpotent Jordan block for $(X_n - a)$ acting on M.

(ii) In the Grothendieck group $K(\mathrm{Rep}\, H_n)$, we have

$$[e_a M] = \varepsilon_a(M)[\widetilde{e}_a M] + \sum k_\alpha [N_\alpha],$$

where N_α are irreducible modules with $\varepsilon_a(N_\alpha) < \varepsilon_a(M) - 1$. (We refer to these N_α as "lower terms".)

(iii) $\varepsilon_a(M) = \dim \mathrm{Hom}_{H_{n-1}}(e_a M, e_a M)$.

2.3. The functor f_a

Write

$$aJ_m = R[x]/((x-a)^m);$$

in other words aJ_m is the H_1-module on which X_1 acts with a-Jordan block of size m, or equivalently $X_1 - a$ acts with nilpotent Jordan block of size m. Sometimes, we will abbreviate aJ_1 as just a.

Notice that if λ_j occurs with multiplicity m in the r-tuple λ, then $\lambda_j J_m$ is an H_1^λ-module, while $\lambda_j J_{m+1}$ is not.

We will use the aJ_m notation for indecomposable H_1-modules; however we will also be referring to nilpotent Jordan blocks when referring to the action of the center of H_n in future sections.

Following [G], define a functor $f_a : \operatorname{Rep} H_n^\lambda \to \operatorname{Rep} H_{n+1}^\lambda$ by

$$f_a N = \varprojlim_m \operatorname{pr}_\lambda \operatorname{Ind}_{H_n \otimes H_1}^{H_{n+1}} N \boxtimes aJ_m. \qquad (*)$$

By the definition of $\operatorname{pr}_\lambda$, if $N \in \operatorname{Rep} H_n^\lambda$ then $f_a N \in \operatorname{Rep} H_{n+1}^\lambda$. (In which case, we should really be writing $f_a N = \varprojlim_m \operatorname{pr}_\lambda \operatorname{Ind}_{n,1}^{n+1} \operatorname{ev}_\lambda^* N \boxtimes aJ_m$.)

If N has central character $s = S_n \cdot (s_1, \ldots, s_n)$ and $f_a N \neq 0$ then $f_a N$ has central character $s + a = S_{n+1} \cdot (s_1, \ldots, s_n, s_{n+1})$. In the decomposition

$$\operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} N \simeq \bigoplus_{a \in R^\times} f_a N,$$

$f_a N$ is *the* direct summand of $\operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} N$ with central character $s + a$. Although this would serve as a definition of f_a, we emphasize that we need Grojnowski's definition $(*)$ for the proofs of this paper.

As with e_a, we list without proof useful facts about f_a.

We remind the reader that f_a is both left and right adjoint to e_a. The limit in the definition of f_a stabilizes.

Let M be an irreducible H_n-module. Define

$$\widetilde{f}_a M = \operatorname{cosoc} \operatorname{Ind}_{n,1}^{n+1} M \boxtimes aJ_1.$$

Recall that if M is irreducible then $\widetilde{f}_a M$ is irreducible. Also $\operatorname{pr}_\lambda \widetilde{f}_a M = \operatorname{cosoc} f_a M \simeq \operatorname{soc} f_a M$ and $\widetilde{e}_a \widetilde{f}_a M \simeq M$ or 0; $\widetilde{f}_a \widetilde{e}_a M \simeq M$ or 0.

Define

$$\varphi_a(M) = \min\{m \mid \operatorname{pr}_\lambda \operatorname{Ind} M \boxtimes aJ_m = f_a M\} \qquad (1)$$

as the stabilization point of the above limit.

We remark that in [G], $\varphi_a(M)$ is defined as

$$\varphi_a(M) = \max\{m \mid f_a^m M \neq 0\} = \max\{m \mid \widetilde{f}_a^m M \neq 0\}, \qquad (2)$$

and it is a major theorem (9.15 from [G]) that (1) and (2) are equal. However, for this paper we do not need that fact.

Recall $f_a M \simeq \check{\mathrm{pr}}_\lambda \check{\mathrm{I}}\mathrm{nd}\, M \boxtimes aJ_{\varphi(M)}$ as well. (This dual statement can be deduced from Grojnowski's result that $f_a M \simeq \check{f}_a M$ where $\check{f}_a M := \varinjlim_m \check{\mathrm{pr}}_\lambda \check{\mathrm{I}}\mathrm{nd}_{n,1}^{n+1} \mathrm{ev}_\lambda^* M \boxtimes aJ_m$, and then by repeating all of the relevant proofs from [G] with arrows reversed.)

We include Theorem 9.15 from [G] below, remarking that in this context the first condition is our definition, and proving its equivalence with (ii) and (iii) are relatively straightforward.

Theorem 2.4. *Let M be an irreducible H_n^λ-module. Then*

(i) $\varphi_a(M)$ is the smallest m for which $f_a M = \mathrm{pr}_\lambda \mathrm{Ind}_{n,1}^{n+1}(M \boxtimes aJ_m)$.

(ii) In the Grothendieck group $K(\mathrm{Rep}\, H_n^\lambda)$,

$$[f_a M] = \varphi_a(M)[\widetilde{f}_a M] + \sum k'_\alpha [N'_\alpha]$$

where N_α are irreducible modules with $\varepsilon_a(N'_\alpha) < \varepsilon_a(M) + 1$.

(iii) $\varphi_a(M) = \dim \mathrm{Hom}_{H_{n+1}}(f_a M, f_a M)$.

Theorems 2.3 and 2.4 show that M occurs with multiplicity $(\varepsilon_a(M) + 1)\varphi_a(M)$ and $(\varphi_a(M) + 1)\varepsilon_a(M)$ in $e_a f_a M$ and $f_a e_a M$ respectively.

3. $e_a M$ AND c_{n-1}

In the next four sections, we will use the center of H_n to filter the modules $e_a M, f_a M, e_a f_a M$, and $f_a e_a M$ and will consider particular irreducible modules that occur as composition factors of them. Multiplication by the center induces maps between these factors, which we describe in terms of nilpotent Jordan blocks.

For the rest of the paper, fix M an irreducible H_n-module, fix $a \in R^\times$, and write $\varepsilon = \varepsilon_a(M)$. Suppose M has central character $s = S_n \cdot (s_1, \ldots, s_n)$. Write

$$c(n, s) = \sum_{j=1}^{n} (X_j - s_j).$$

Let $c_n = c(n, s)$, $c_{n-1} = c(n-1, s-a)$, $c_{n+1} = c(n+1, s+a)$, noting that $c_k \in Z(H_k)$ and

$$c_{n-1} + (X_n - a) = c_n \tag{3}$$

$$c_n + (X_{n+1} - a) = c_{n+1}. \tag{4}$$

We will use the c_k to define and describe filtrations on the modules $e_a M$, $f_a M$, $e_a f_a M$, and $f_a e_a M$.

When discussing a filtration $\{\mathcal{F}_j\}$, we will often refer to $\mathcal{F}_j/\mathcal{F}_{j+1}$ as its j^{th} step. If an irreducible module N occurs with multiplicity k as a composition factor of $\mathcal{F}_j/\mathcal{F}_{j+1}$, we will say there are k copies of N in the j^{th} step.

Claim 3.1. *$c_n M = 0$; $\qquad c_{n-1}\widetilde{e}_a M = 0$.*

Proof. Because M is irreducible, the center of H_n acts by scalars on M, so by definition of central character, $c_n M = 0$. As the central character of $\widetilde{e}_a M$, if it is nonzero, is $s - a$, the same argument gives $c_{n-1}\widetilde{e}_a M = 0$.

Claim 3.2. *$c_{n-1}^{\varepsilon} e_a M = 0$; if $\varepsilon \neq 0$ then $c_{n-1}^{\varepsilon-1} e_a M \neq 0$.*

Proof. By (3) $c_{n-1}M = (c_n - (X_n - a))\, M = -(X_n - a)M$ and so we could have equivalently defined $e_a M$ as the generalized 0-eigenspace of c_{n-1} on $\mathrm{Res}^n_{n-1,1} M$. That the maximal Jordan-block size is $\varepsilon = \varepsilon_a(M)$ follows from Theorem 2.3.

As c_k is central in H_k, left multiplication by c_k on any module induces an endomorphism which, by abuse of notation, we'll also denote c_k.

The images of c_{n-1}^j give the following filtration of $e_a M$

$$e_a M = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_\varepsilon = 0 \tag{5}$$

where $A_j = c_{n-1}^j e_a M = \mathrm{Im}\, c_{n-1}^j$.

Claim 3.3. *$\mathrm{cosoc}\, A_j/A_{j+1} \simeq \widetilde{e}_a M$ and $\widetilde{e}_a M$ occurs with multiplicity one as a composition factor of A_j/A_{j+1}.*

Proof. First, notice that $A_j \neq A_{j+1}$ as c_{n-1} is nilpotent on $e_a M$ and $c_{n-1}^{\varepsilon-1} e_a M \neq 0$. Consider the map

$$e_a M \xrightarrow{c_{n-1}^j} A_j \twoheadrightarrow A_j/A_{j+1}.$$

Since the composition is surjective and $\widetilde{e}_a M \simeq \mathrm{cosoc}\, M$, it follows that $\mathrm{cosoc}\, A_j/A_{j+1} \simeq \widetilde{e}_a M$. Hence there is at least one copy of $\widetilde{e}_a M$ in each of the ε steps of the filtration (5). However, by Theorem 2.3, $\widetilde{e}_a M$ occurs with multiplicity ε as a composition factor of $e_a M$, so there must be exactly one copy of $e_a M$ at each step of filtration.

Although informal, it is unambiguous to talk about the ε copies of $\widetilde{e}_a M$ in $e_a M$ as the order of vanishing of c_{n-1} distinguishes them. In fact, multiplication by c_{n-1} carries one into the next, and in this sense the copies of $\widetilde{e}_a M$ sit in $e_a M$ uniserially.

Claim 3.4. *$A_{\varepsilon-1} = \widetilde{e}_a M$.*

Proof. As $\widetilde{e}_a M = \mathrm{soc}\, e_a M$, we must have $\widetilde{e}_a M \subseteq A_{\varepsilon-1}$ and therefore $\mathrm{soc}\, A_{\varepsilon-1} = \widetilde{e}_a M$. On the other hand, by claim 3.3 above, since $A_\varepsilon = 0$, $\mathrm{cosoc}\, A_{\varepsilon-1} \simeq \widetilde{e}_a M$ and $\widetilde{e}_a M$ occurs with multiplicity one in $A_{\varepsilon-1}$. We must have $A_{\varepsilon-1} = \widetilde{e}_a M$.

Below we extend Theorem 2.3(iii).

Claim 3.5. $\mathrm{Hom}_{H_{n-1}}(e_aM, e_aM) \simeq R[x]/(x^\varepsilon)$ *via the map* $c_{n-1} \mapsto \bar{x}$. *In particular, this shows* $\dim_R \mathrm{Hom}_{H_{n-1}}(e_aM, e_aM) = \varepsilon$.

Proof. We will downward induct on j upon the statement $\{c_{n-1}^k\}_{\varepsilon > k \geq j}$ is a basis of $\mathrm{Hom}_{H_{n-1}}(e_aM, A_j)$. Their linear independence follows from claim 3.2, so we need only show they span the space. In the case $j = \varepsilon - 1$, because $\widetilde{e}_aM \simeq \mathrm{cosoc}\, e_aM$, we see that $\mathrm{Hom}_{H_{n-1}}(e_aM, A_{\varepsilon-1}) = \mathrm{Hom}_{H_{n-1}}(e_aM, \widetilde{e}_aM) = R$. As $c_{n-1}^{\varepsilon-1}$ is a nonzero map $e_aM \to \widetilde{e}_aM$, this map must generate the space above, i.e. $\{c_{n-1}^k\}_{\varepsilon > k \geq \varepsilon - 1}$ is a basis.

Now let $0 \neq \alpha \in \mathrm{Hom}_{H_{n-1}}(e_aM, A_j)$. We may assume $\alpha(e_aM) \not\subseteq A_{j+1}$. Thus $e_aM \xrightarrow{\alpha} A_j \to A_j/A_{j+1}$ is a nonzero map and so its image contains a copy of $\widetilde{e}_aM \simeq \mathrm{cosoc}\, e_aM$. Since $\widetilde{e}_aM$ occurs with multiplicity one in A_j/A_{j+1}, the composition

$$e_aM \xrightarrow{\alpha} A_j \twoheadrightarrow A_j/A_{j+1} \twoheadrightarrow \widetilde{e}_aM \tag{6}$$

is a nonzero map. The map α must be a surjection because $\widetilde{e}_aM = \mathrm{cosoc}\, e_aM = \mathrm{cosoc}\, A_j$. Hence $c_{n-1}^{\varepsilon-j-1} \circ \alpha \neq 0$. Further, $c_{n-1}^{\varepsilon-j-1} \circ \alpha \in \mathrm{Hom}_{H_{n-1}}(e_aM, A_{\varepsilon-1})$ so $\exists b \in R$ such that $c_{n-1}^{\varepsilon-j-1} \circ \alpha = bc_{n-1}^{\varepsilon-1}$, in other words $c_{n-1}^{\varepsilon-j-1} \circ (\alpha - bc_{n-1}^j) = 0$. Applying the contrapositive of the above argument to $\alpha - bc_{n-1}^j$, we see the image of $\alpha - bc_{n-1}^j$ must be contained in A_{j+1}. By the inductive hypothesis, $\alpha - bc_{n-1}^j = \sum_{k>j} b_k c_{n-1}^k$ for some $b_k \in R$, and so $\{c_{n-1}^k\}_{\varepsilon > k \geq j}$ forms a basis of $\mathrm{Hom}_{H_{n-1}}(e_aM, A_j)$. The case $j = 0$ gives the claim since $A_0 = e_aM$.

Remark 3.6. *Claim 3.5 shows that* c_{n-1} *acts on the* ε *copies of* $\widetilde{e}_aM$ *in* e_aM *as* x *does on* $R[x]/(x^\varepsilon)$*, in other words, like a nilpotent Jordan block of size* ε.

Corollary 3.7. $\dim \mathrm{Hom}_{H_n}(\mathrm{Ind}_{n-1,1}^n e_aM \boxtimes aJ_m, M) = \min(\varepsilon, m)$.

Proof. By Frobenius reciprocity

$$\begin{aligned}\mathrm{Hom}_{H_n}(\mathrm{Ind}\, e_aM \boxtimes aJ_m, M) &= \mathrm{Hom}_{H_{n-1}\otimes H_1}(e_aM \boxtimes aJ_m, \mathrm{Res}_{n-1,1}^n M) \\ &= \mathrm{Hom}_{H_{n-1}\otimes H_1}(e_aM \boxtimes aJ_m, \Delta_aM).\end{aligned}$$

Let $\beta \in \mathrm{Hom}_{H_{n-1}\otimes H_1}(e_aM \boxtimes aJ_m, \Delta_aM)$. Then β determines a map $\bar{\beta} \in \mathrm{Hom}_{H_{n-1}}(e_aM, e_aM)$ by $\bar{\beta}(u) = \beta(u \otimes 1)$. By claim 3.1 and equation (3), $(a - X_n)$ acts on Δ_aM exactly as c_{n-1} acts on Δ_aM (or e_aM). Because $(X_n - a)^m(aJ_m) = 0$, we see

$$\begin{aligned} c_{n-1}^m\bar{\beta}(u) &= c_{n-1}^m\beta(u \otimes 1) = (a - X_n)^m\beta(u \otimes 1) \\ &= \beta(u \otimes (a - X_n)^m) = \beta(u \otimes 0) = 0. \end{aligned}$$

Thus, by claim 3.5 we can write $\bar{\beta} = \sum_{k=0}^{\varepsilon-1} b_k c_{n-1}^k$, where $b_k = 0$ if $k < \varepsilon - m$. The space of such $\bar{\beta}$ clearly has dimension $\min(\varepsilon, m)$.

Conversely, any $\bar{\beta} \in \mathrm{Hom}_{H_{n-1}}(e_a M, e_a M)$ such that $c_{n-1}^m \bar{\beta} = 0$ determines a map $\beta \in \mathrm{Hom}_{H_{n-1} \otimes H_1}(e_a M \boxtimes aJ_m, \Delta_a M)$ via $\beta(u \otimes (X_n - a)^k) = (-c_{n-1})^k \bar{\beta}(u) = \bar{\beta}((-c_{n-1})^k u)$. Thus the space of such β also has dimension $\min(\varepsilon, m)$.

Observe in the proof above we have used the fact $c_n M = 0$ to recover $\Delta_a M$ from $e_a M$ and thus construct β from $\bar{\beta}$. For a more careful treatment of the distinction between Δ_a and e_a, see [GV].

We can also dualize the above discussion considering the filtration

$$0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\varepsilon = e_a M \tag{7}$$

where $B_j = \ker c_{n-1}^j$.

The following claims and proofs are dual to 3.3 and 3.4, but we include them for completeness.

Observe $B_j \supseteq A_{\varepsilon - j}$ as $c_{n-1}^j (c_{n-1}^{\varepsilon - j} e_a M) = c_{n-1}^\varepsilon e_a M = 0$.

Claim 3.8. *$\mathrm{soc}\, B_j / B_{j-1} \simeq \widetilde{e}_a M$ and $\widetilde{e}_a M$ occurs with multiplicity one as a composition factor of B_j / B_{j-1}.*

Proof. First, $B_j / B_{j-1} \neq 0$ for $0 < j \leq \varepsilon$ since $c_{n-1}^{\varepsilon-1} e_a M \neq 0$. Consider $B_j \xrightarrow{c_{n-1}^{j-1}} B_1$, which is nonzero and well-defined because $c_{n-1}^{j-1} B_j \neq 0$, but $c_{n-1}(c_{n-1}^{j-1} B_j) = 0$. The kernel of the above map is exactly B_{j-1} by definition, and so we have an injection

$$B_j / B_{j-1} \hookrightarrow B_1.$$

Thus $\mathrm{soc}\, B_j / B_{j-1} \hookrightarrow \mathrm{soc}\, B_1 \subseteq \mathrm{soc}\, e_a M = \widetilde{e}_a M$, giving $\mathrm{soc}\, B_j / B_{j-1} \simeq \widetilde{e}_a M$. Again, as there are only ε copies of $\widetilde{e}_a M$ in $e_a M$, there can be only one copy of $\widetilde{e}_a M$ in each B_j / B_{j-1}.

Claim 3.9. *$e_a M / B_{\varepsilon - 1} \simeq \widetilde{e}_a M$.*

Proof. $\widetilde{e}_a M \simeq \mathrm{cosoc}\, e_a M = \mathrm{cosoc}\, e_a M / B_{\varepsilon-1}$, but also by claim 3.8 $\widetilde{e}_a M \simeq \mathrm{soc}\, e_a M / B_{\varepsilon - 1}$ which occurs with multiplicity one, thus yielding $e_a M / B_{\varepsilon-1} \simeq \widetilde{e}_a M$.

Remark 3.10. *As c_{n-1} annihilates semisimple H_{n-1}-modules with central character $s - a$, we see the cosocle filtration of $e_a M$ must refine the filtration (5), while its socle filtration refines (7). In any case, the copies of $\widetilde{e}_a M$ must all lie in different steps of these filtrations, although we have little control over the other composition factors.*

4. f_aM AND c_{n+1}

In the following sections, we fix λ and assume M is an irreducible H_n^λ-module with central character s. However, we will also be considering M as an H_n-module, but be sloppy about denoting it $\mathrm{ev}_\lambda^* M$. However, it is necessary to have fixed λ to define f_a and φ_a. Write $\varphi = \varphi_a(M)$.

As the central character of f_aM, if it is nonzero, is $s + a$, c_{n+1} acts nilpotently on f_aM.

Claim 4.1. *$c_{n+1}^{\varphi} f_a M = 0$; if $\varphi \neq 0$ then $c_{n+1}^{\varphi-1} f_a M \neq 0$.*

We include the proof with that of claim 4.2 below.

Again, we can make filtrations by the images and kernels of c_{n+1}^j:

$$f_aM = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_\varphi = 0, \tag{8}$$
$$0 = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_\varphi = f_aM, \tag{9}$$

where $C_j = \operatorname{Im} c_{n+1}^j$ and $D_j = \ker c_{n+1}^j$.

Claim 4.2. *There is exactly one copy of $\widetilde{f}_aM$ in C_j/C_{j+1} and D_j/D_{j-1} occurring in the cosocle and socle, respectively.*

Proof. We begin by calculating

$$\begin{aligned}
c_{n+1} \operatorname{Ind}_{n,1}^{n+1} M \boxtimes aJ_m =& c_{n+1} H_{n+1} \bigotimes_{H_n \otimes H_1} M \boxtimes aJ_m \\
=& H_{n+1} \bigotimes_{H_n \otimes H_1} c_{n+1} M \boxtimes aJ_m \\
=& H_{n+1} \bigotimes_{H_n \otimes H_1} (c_n + (X_{n+1} - a)) M \boxtimes aJ_m \\
=& H_{n+1} \bigotimes_{H_n \otimes H_1} (c_n M \boxtimes aJ_m + M \boxtimes (X_{n+1} - a) aJ_m) \\
=& 0 + H_{n+1} \bigotimes_{H_n \otimes H_1} M \boxtimes aJ_{m-1} \\
=& \operatorname{Ind}_{n,1}^{n+1} M \boxtimes aJ_{m-1}.
\end{aligned}$$

Consequently, $c_{n+1}^{\varphi} \operatorname{Ind} M \boxtimes aJ_\varphi = 0$, but $c_{n+1}^{\varphi-1} \operatorname{Ind} M \boxtimes aJ_\varphi \neq 0$ and c_{n+1} induces the following filtration:

$$\operatorname{Ind} M \boxtimes aJ_\varphi \supseteq \operatorname{Ind} M \boxtimes aJ_{\varphi-1} \supseteq \cdots \supseteq \operatorname{Ind} M \boxtimes aJ_1 \supseteq 0, \tag{10}$$

where the j^{th} step is $\operatorname{Im} c_{n+1}^j = \ker c_{n+1}^{\varepsilon - j}$.

Because $\operatorname{Ind} M \boxtimes aJ_m / \operatorname{Ind} M \boxtimes aJ_{m-1} \simeq \operatorname{Ind} M \boxtimes aJ_1$ and

$$[\operatorname{Ind} M \boxtimes aJ_1] = [\widetilde{f}_a M] + \text{lower terms}, \tag{11}$$

there is exactly one copy of $\widetilde{f}_a M$ in each step of filtration (10) above, which occurs in the cosocle (because by definition $\widetilde{f}_a M = \operatorname{cosoc} \operatorname{Ind} M \boxtimes aJ_1$). Thus there are a total of φ copies of $\widetilde{f}_a M$ occurring in $\operatorname{Ind} M \boxtimes aJ_\varphi$.

Returning to $f_a M = \operatorname{pr}_\lambda \operatorname{Ind} M \boxtimes aJ_\varphi$, because φ is defined here as the "stabilization point", this means precisely that there are φ copies of $\widetilde{f}_a M$ in $f_a M$; in other words, $\operatorname{pr}_\lambda$ does not kill any of the φ copies of $\widetilde{f}_a M$ in $\operatorname{Ind} M \boxtimes aJ_\varphi$. The calculation above shows that $\operatorname{Ind} M \boxtimes aJ_{\varphi-j}$ surjects to C_j which in turn surjects to C_j/C_{j+1}, showing that $\widetilde{f}_a M$ is in $\operatorname{cosoc} C_j/C_{j+1}$. Then, by counting, there is exactly one copy of $\widetilde{f}_a M$ in C_j/C_{j+1}.

However, to see that $\operatorname{soc} D_j/D_{j-1} = \widetilde{f}_a M$, we use the fact $\operatorname{soc} f_a M = \widetilde{f}_a M$ and argue as in claim 3.8. (Alternatively, we can make a filtration using $\check{\operatorname{Ind}}$, the right adjoint of Res, using the fact $f_a M = \check{\operatorname{pr}}_\lambda \check{\operatorname{Ind}} M \boxtimes aJ_\varphi$.)

Remark 4.3. *We have shown the action of c_{n+1} on the φ copies of $\widetilde{f}_a M$ in $f_a M$ is exactly the action of $X_{n+1} - a$ on aJ_φ, in other words of a nilpotent Jordan block of size φ.*

As with claims 3.4 and 3.9, the following two claims hold. We omit the proofs as they are similar.

Claim 4.4. *$C_{\varphi-1} \simeq \widetilde{f}_a M$; $f_a M/D_{\varphi-1} = \widetilde{f}_a M$.*

Claim 4.5. $\operatorname{Hom}_{H_{n+1}}(f_a M, f_a M) \simeq R[x]/(x^\varphi)$ *via the map $c_{n+1} \mapsto \bar{x}$. In particular,* $\dim \operatorname{Hom}_{H_{n+1}}(f_a M, f_a M) = \varphi$.

5. $e_a f_a M$ AND c_n

Next we will study the interactions between e_a and f_a.

Write

$$kM = \bigoplus_{j=1}^{k} M$$

for the direct sum of k copies of M.

Remark 5.1. *The cyclotomic Mackey decomposition is the following isomorphism,*

$$rM \bigoplus \operatorname{Ind}_{H_{n-1}^\lambda}^{H_n^\lambda} \operatorname{Res}_{H_{n-1}^\lambda}^{H_n^\lambda} M \simeq \operatorname{Res}_{H_n^\lambda}^{H_{n+1}^\lambda} \operatorname{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda} M$$

where $r = \sum \lambda_i$. This isomorphism is given by the map the map $(0, 1 \otimes u) \mapsto T_n \otimes u$ and $(u^{(j)}, 0) \mapsto X_{n+1}^j \otimes u$ where $u \in M$ and $u^{(j)}$ denotes $(0, \ldots, 0, u, 0, \ldots, 0)$ at the j^{th} copy of M in $rM = \bigoplus_{j=1}^{r} M$, and we leave it *to the reader to verify this is an isomorphism. Comparing central characters yields*

$$rM \bigoplus_a f_a e_a M \simeq \bigoplus_a e_a f_a M$$

(and also yields that $f_a e_b M \simeq e_b f_a M$ if $a \neq b$). In sections 6 and 7 we will give a more explicit description of the images of $f_a e_a M$ and $e_a f_a M$ under the isomorphism above. When we restrict the map to $f_a e_a M$, we always get either an injection or surjection to $e_a f_a M$ and we will show the cokernel or kernel is a direct sum of $|\varphi_a - \varepsilon_a|$ copies of M.

Below we compute the socle (and cosocle) of $e_a f_a M$ and $f_a e_a M$. Then we will describe the action of c_n on the copies of M in $e_a f_a M/\operatorname{soc}(e_a f_a M)$, and use that to eventually describe the action on $e_a f_a M$ and $f_a e_a M$.

Claim 5.2. $\operatorname{soc} f_a e_a M \simeq \operatorname{cosoc} f_a e_a M = \varepsilon M$;
$\operatorname{soc} e_a f_a M \simeq \operatorname{cosoc} e_a f_a M = \varphi M$

Proof. By the adjointness of f_a and e_a and by claim 3.5, we have $\varepsilon = \dim \operatorname{Hom}(e_a M, e_a M) = \dim \operatorname{Hom}(f_a e_a M, M) = \dim \operatorname{Hom}(M, f_a e_a M)$, showing M has multiplicity ε in the socle and cosocle. We just need to show no other irreducibles can occur.

Let N be an irreducible H_n-module and $\gamma \in \operatorname{Hom}_{H_{n-1}}(e_a M, e_a N) \simeq \operatorname{Hom}_{H_n}(f_a e_a M, N)$. If $\gamma \neq 0$, it has a copy of $\operatorname{cosoc} e_a M = \widetilde{e}_a M$ in its image. By Theorem 2.3, the composition factors of $e_a N$ are modules with $\varepsilon_a \leq \varepsilon_a(\widetilde{e}_a N) = \varepsilon_a(N) - 1$. This shows $\varepsilon_a(\widetilde{e}_a M) \leq \varepsilon_a(\widetilde{e}_a N)$. On the other hand, $\operatorname{soc} e_a N = \widetilde{e}_a N \subseteq \operatorname{Im} \gamma$ so that $e_a M$ has $\widetilde{e}_a N$ as a subquotient, yielding $\varepsilon_a(\widetilde{e}_a N) \leq \varepsilon_a(\widetilde{e}_a M)$. As $\widetilde{e}_a M$ is the only composition factor of $e_a M$ with $\varepsilon_a = \varepsilon_a(M) - 1 = \varepsilon_a(N) - 1$, and similarly for $\widetilde{e}_a N$, γ must map them to each other, yielding $\widetilde{e}_a M \simeq \widetilde{e}_a N$ and so $M \simeq N$.

The proof for $e_a f_a M$ is similar.

Remark 5.3. *Here we make more precise the notion of "c_n acting on copies of M". This description also applies to remarks 3.6 and 4.3.*

Suppose we have an H_n^λ-module $\mathcal{C}$ with central character s, such that $c_n^k \mathcal{C} = 0$ for $k \gg 0$. Let $0 \subseteq \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ be a filtration in which M occurs as a composition factor of $\mathcal{B}/\mathcal{A}$ with multiplicity one. Since we have a surjection $\mathcal{B}/\mathcal{A} \twoheadrightarrow c_n\mathcal{B}/c_n\mathcal{A}$, M occurs at most once as a composition factor of $c_n\mathcal{B}/c_n\mathcal{A}$. If M does occur once in $c_n\mathcal{B}/c_n\mathcal{A}$, we shall say that c_n carried the copy of M in $\mathcal{B}/\mathcal{A}$ to the copy in $c_n\mathcal{B}/c_n\mathcal{A}$, and that otherwise it is sent to 0.

Suppose we now had a composition series of $\mathcal{C}$

$$0 = \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_t = \mathcal{C}$$

that was compatible with c_n in the sense that if $\mathcal{C}_i$ occurs in the filtration, so does $c_n\mathcal{C}_i$. Then we can consider all subquotients $\mathcal{C}_i/\mathcal{C}_{i-1} \simeq M$ that are not of the form $c_n\mathcal{C}_j/c_n\mathcal{C}_{j-1}$ for some $j > i$, and try to record which *copies $\mathcal{C}_k/\mathcal{C}_{k-1} \simeq M$ they are carried to by repeated action of c_n, eventually accounting for each composition factor isomorphic to M. If the filtration is compatible with the kernels of c_n (as well as images), then we will account for each copy exactly once. (The filtrations considered below have these nice properties.) Such a chain of subquotients corresponds to a Jordan block. This is what we refer to as describing the Jordan block structure of the action of c_n on the copies of M in $\mathcal{C}$.*

To understand $e_a f_a M$, we first need to understand $e_a f_a M/\operatorname{soc}(e_a f_a M)$. We will show c_n acts on the copies of M in each of the above modules as it does on $e_a \operatorname{Ind} M \boxtimes aJ_\varphi$ and $\operatorname{Ind} e_a M \boxtimes aJ_\varphi$ respectively. Further, the action of c_n on the latter of each pair is as $x\otimes 1+1\otimes y$ acts on $R[x]/(x^\varepsilon)\boxtimes R[y]/(y^\varphi)$. Thus we include the following proposition, but omit the proof as it is an exercise in linear algebra.

Proposition 5.4. *The action of $x\otimes 1 + 1\otimes y$ on $R[x]/(x^\varepsilon)\boxtimes R[y]/(y^\varphi)$ is*

(i) as ε Jordan blocks of sizes $\varphi+\varepsilon-1, \varphi+\varepsilon-3, \dots, \varphi-\varepsilon+1$, if $\varepsilon\leq\varphi$ and $\operatorname{char} R = 0$;

(ii) as φ Jordan blocks of sizes $\varepsilon+\varphi-1, \varepsilon+\varphi-3, \dots, \varepsilon-\varphi+1$, if $\varphi\leq\varepsilon$ and $\operatorname{char} R = 0$.

(iii) as $\min(\varepsilon,\varphi)$ Jordan blocks, which have sizes that range from $\varepsilon+\varphi-1$ to $\varepsilon+\varphi+1-2\min(\varepsilon,\varphi)$ if $\operatorname{char} R = p > 0$.

In particular, in part (iii) of proposition 5.4, the sizes of the Jordan blocks need not jump by 2, although they do achieve the maximal and minimal sizes. As it is not used here and is rather complicated, we do not list the actual sizes of the blocks.

First, consider $\operatorname{Ind} e_a M\boxtimes aJ_\varphi$. It contains $\varepsilon\varphi$ copies of M by part (ii) of Theorem 2.3 and has submodules $G_{j,k} = \operatorname{Ind} A_j \boxtimes aJ_k$ which each contain $(\varepsilon - j)k$ copies of M. We compute:

$$\begin{aligned}
c_n G_{j,k} =& c_n \operatorname{Ind} A_j \boxtimes aJ_k\\
=& c_n H_n \bigotimes_{H_{n-1}\otimes H_1} A_j \boxtimes aJ_k\\
=& H_n \bigotimes_{H_{n-1}\otimes H_1} c_n A_j \boxtimes aJ_k\\
=& H_n \bigotimes_{H_{n-1}\otimes H_1} c_{n-1} A_j \boxtimes aJ_k + H_n \bigotimes_{H_{n-1}\otimes H_1} A_j \boxtimes (X_n - a) aJ_k\\
=& H_n \bigotimes_{H_{n-1}\otimes H_1} A_{j+1} \boxtimes aJ_k + H_n \bigotimes_{H_{n-1}\otimes H_1} A_j \boxtimes aJ_{k-1}\\
=& G_{j+1,k} + G_{j,k-1}.
\end{aligned}$$

So the action of c_n on the $\varepsilon\varphi$ copies of M in $\operatorname{Ind} e_a M \boxtimes aJ_\varphi$ is exactly the action of

$$c_{n-1} \otimes 1 + 1 \otimes (X_n - a)$$

on $e_a M \boxtimes aJ_\varphi$. In other words, c_n acts as $x \otimes 1 + 1 \otimes y$ acts on $R[x]/(x^\varepsilon) \boxtimes R[y]/(y^\varphi)$.

Lemma 5.5. *The Jordan block structure describing the action of c_n on the $\varepsilon\varphi$ copies of M in $e_a f_a M/\operatorname{soc}(e_a f_a M)$ is that of $x \otimes 1 + 1 \otimes y$ acting on $R[x]/(x^\varepsilon) \boxtimes R[y]/(y^\varphi)$.*

Proof. First, we have a surjection $\operatorname{Ind} M \boxtimes aJ_\varphi \overset{\mathrm{pr}_\lambda}{\twoheadrightarrow} f_a M$, such that the composition factors of the kernel of pr_λ consists of modules N with $\varepsilon_a(N) < \varepsilon + 1 = \varepsilon_a(\widetilde{f}_a M)$, and so in particular the kernel contains no copies of $\widetilde{f}_a M$. Applying the exact functor e_a, we see the kernel of $e_a \operatorname{Ind} M \boxtimes aJ_\varphi \twoheadrightarrow e_a f_a M$ consists of modules with $\varepsilon_a < \varepsilon$, and in particular contains no copies of M. This also implies that the structure of the copies of $\widetilde{f}_a M$ in $\operatorname{Ind} M \boxtimes aJ_\varphi$ is the same as that in $f_a M$ and that the structure of the copies of M in $e_a \operatorname{Ind} M \boxtimes aJ_\varphi$ is exactly the same as that in $e_a f_a M$. The affine Mackey decomposition (see [GV]) gives us

$$0 \longrightarrow \varphi M \longrightarrow e_a \operatorname{Ind} M \boxtimes aJ_\varphi \longrightarrow \operatorname{Ind} e_a M \boxtimes aJ_\varphi \longrightarrow 0 \qquad (12)$$

where the kernel is exactly the H_n-submodule $1 \otimes (M \boxtimes aJ_\varphi) \simeq \operatorname{Ind}_n^n M \boxtimes e_a(aJ_\varphi) \simeq \varphi M$. As the map $e_a \operatorname{Ind} M \boxtimes aJ_\varphi \twoheadrightarrow e_a f_a M$ has no copies of M in its kernel, the kernel of the exact sequence (12) is mapped exactly to the socle of $e_a f_a M$, which we saw was equal to φM in claim 5.2. Therefore we have a surjection

$$\operatorname{Ind} e_a M \boxtimes aJ_\varphi \twoheadrightarrow e_a f_a M/\operatorname{soc}(e_a f_a M). \qquad (13)$$

By counting, there cannot be any copies of M in the kernel of this map. Thus the action of c_n on the copies of M in $e_a f_a M/\operatorname{soc}(e_a f_a M)$ is exactly as described above for $\operatorname{Ind} e_a M \boxtimes aJ_\varphi$.

From the above discussion, it follows that:

Corollary 5.6. $\operatorname{cosoc}(e_a f_a M/\operatorname{soc}(e_a f_a M)) = \min(\varepsilon, \varphi)\ M$.

Next, we will use lemma 5.5 to compute the Jordan block structure of c_n acting on the copies on M in $e_a f_a M$. As in the previous sections, we create filtrations

$$e_a f_a M = E_0 \supseteq E_1 \supseteq \cdots \supseteq E_{\varphi+\varepsilon} = 0 \qquad (14)$$

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{\varphi+\varepsilon} = e_a f_a M \qquad (15)$$

where $E_j = c_n^j e_a f_a M = \operatorname{Im} c_n^j$ and $F_j = \ker c_n^j$.

The length of each filtration is either $\varphi + \varepsilon$ or $\varphi + \varepsilon - 1$ because $c_n \operatorname{soc}(e_a f_a M) = 0$ and by lemma 5.5 $c_n^{\varphi+\varepsilon-1}(e_a f_a M/\operatorname{soc} e_a f_a M) = 0$, but

$c_n^{\varphi+\varepsilon-2}(e_a f_a M/\operatorname{soc} e_a f_a M) \neq 0$. Below we will see the length is actually $\varphi+\varepsilon$.

We have slightly more complicated proofs of the analogous statements to claims 3.3, 3.8, and 4.2.

Lemma 5.7. *The only copies of M in E_j/E_{j+1} and F_j/F_{j-1} occur in the cosocle and socle respectively. In particular, c_n kills a copy of M if and only if it lies in the socle of $e_a f_a M$; a copy of M is outside of $c_n e_a f_a M$ if and only if it occurs in the cosocle.*

Proof. As $e_a f_a M/c_n e_a f_a M = E_0/E_1 \overset{c_n^j}{\twoheadrightarrow} E_j/E_{j+1}$, it suffices to prove the statement for $e_a f_a M/c_n e_a f_a M$. Because c_n annihilates simple modules with central character s, $\operatorname{cosoc}(E_0/E_1) = \operatorname{cosoc}(e_a f_a M/c_n e_a f_a M) = \operatorname{cosoc}(e_a f_a M) = \varphi M$. We will show no other copies of M occur in E_0/E_1.

Consider the exact sequence

$$0 \longrightarrow c_n e_a f_a M + \operatorname{soc} e_a f_a M \Big/ c_n e_a f_a M \longrightarrow e_a f_a M \Big/ c_n e_a f_a M \longrightarrow \\ (e_a f_a M/\operatorname{soc} e_a f_a M) \Big/ c_n(e_a f_a M/\operatorname{soc} e_a f_a M) \longrightarrow 0. \quad (16)$$

By lemma 5.5 and proposition 5.4, there are $\min(\varepsilon,\varphi)$ copies of M in the image of the above exact sequence (one for each Jordan block). Because c_n kills simple modules, the cosocle of the image above is equal to $\operatorname{cosoc}(e_a f_a M/\operatorname{soc} e_a f_a M)$, which is $\min(\varepsilon,\varphi)M$ by corollary 5.6. That is to say, all the $\min(\varepsilon,\varphi)$ copies of M occurring in the image of (16) occur in its cosocle.

We will next describe the copies of M in $c_n e_a f_a M \cap \operatorname{soc} e_a f_a M$, noting the kernel of (16) is isomorphic to $c_n e_a f_a M/(c_n e_a f_a M \cap \operatorname{soc} e_a f_a M)$.

To do this, we use the surjection (13) from lemma 5.5, recalling it maps $\varphi M \simeq 1 \otimes M \boxtimes aJ_\varphi \subseteq e_a \operatorname{Ind} M \boxtimes aJ_\varphi$ exactly to $\operatorname{soc} e_a f_a M$. We shall show $1 \otimes M \boxtimes aJ_{\min(\varepsilon,\varphi)} \subseteq c_n(e_a \operatorname{Ind} M \boxtimes aJ_\varphi)$ by direct computation.

Pick $u \in M$ such that $c_{n-1}^{\varepsilon-1}u \neq 0$ but $c_{n-1}^{\varepsilon}u = 0$ (i.e. $u \in A_0 \setminus A_1$). As M is irreducible, $H_n c_{n-1}^j u = M$ for $0 \le j < \varepsilon$.

Write v_j for $(X_{n+1}-a)^j \in aJ_\varphi$ so that $aJ_\varphi = \operatorname{span}\{v_0,\ldots,v_{\varphi-1}\}$. For $1 \le j \le \min(\varepsilon,\varphi)$ let

$$w_j = c_{n-1}^{\varepsilon-j}u \otimes v_{\varphi-1} - c_{n-1}^{\varepsilon-j+1}u \otimes v_{\varphi-2} + \cdots \pm c_{n-1}^{\varepsilon-1}u \otimes v_{\varphi-j}.$$

Notice $c_{n-1}w_j = w_{j-1}$ and $(X_{n+1}-a)w_j = -w_{j-1}$.

Then

$$\begin{aligned} c_n(T_n \otimes w_j) &= (c_{n-1} + (X_n - a))T_n \otimes w_j \\ &= T_n \otimes c_{n-1}w_j + T_n \otimes ((X_{n+1}-a)w_j) + 1 \otimes (1-q)X_{n+1}w_j \\ &= 1 \otimes (1-q)X_{n+1}w_j \in 1 \otimes M \boxtimes aJ_\varphi. \end{aligned}$$

As j ranges, the $1 \otimes X_{n+1} w_j$ generate $1 \otimes M \boxtimes aJ_{\min(\varepsilon,\varphi)}$ over H_n, yielding $1 \otimes M \boxtimes aJ_{\min(\varepsilon,\varphi)} \subseteq c_n(e_a \operatorname{Ind} M \boxtimes aJ_\varphi)$.

Applying the surjection (13) shows $\min(\varepsilon,\varphi)M \subseteq c_n e_a f_a M$.

Next, we'll show $c_n e_a f_a M \cap \operatorname{soc} e_a f_a M = \min(\varepsilon,\varphi)M$.

Let $Q \subseteq e_a f_a M$ be such that $e_a f_a M/Q = \operatorname{cosoc} e_a f_a M \simeq \varphi M$. The fact $c_n M = 0$ yields $c_n e_a f_a M \subseteq Q$, so that the direct sum $\min(\varepsilon,\varphi)M \subseteq c_n e_a f_a M \subseteq Q$. Also $\operatorname{cosoc}(e_a f_a M/\operatorname{soc} e_a f_a M) = \min(\varepsilon,\varphi)M$. Recalling $\operatorname{cosoc} e_a f_a M = \varphi M$, we must have that $\varphi - \min(\varepsilon,\varphi)$ copies of M that lie in the socle are also in the cosocle, i.e. have trivial intersection with Q and therefore with $c_n e_a f_a M$. This implies that $\operatorname{soc} Q = \min(\varepsilon,\varphi)M$ and that $\operatorname{soc}(c_n e_a f_a M) = \min(\varepsilon,\varphi)M = c_n e_a f_a M \cap \operatorname{soc}(e_a f_a M)$.

In the case $\varphi > \varepsilon$, the kernel of (16) is isomorphic to

$$\operatorname{soc} e_a f_a M/(\operatorname{soc} e_a f_a M \cap c_n e_a f_a M) = \varphi M/\min(\varepsilon,\varphi)M = (\varphi-\varepsilon)M.$$

However, these are the copies of M having trivial intersection with Q, in other words they do occur in $\operatorname{cosoc} e_a f_a M$. (This shows $(\varphi-\varepsilon)M$ is a direct summand of $e_a f_a M$. Later we shall see its direct complement is $f_a e_a M$.)

In the case $\varphi \le \varepsilon$, the kernel of (16) is 0, showing that in either case, there are exactly φ copies of M in $e_a f_a M/c_n e_a f_a M$ all of which must therefore occur in the cosocle.

The proof for F_j/F_{j-1} is similar and leads to the conclusion that the only copies of M in the kernel of c_n are exactly the φ copies of M in $\operatorname{soc} e_a f_a M$.

Theorem 5.8. *The action of c_n on the $(\varepsilon+1)\varphi$ copies of M in $e_a f_a M$ is*

(i) as φ Jordan blocks, ε of which have sizes $\varphi+\varepsilon, \varphi+\varepsilon-2, \ldots, \varphi-\varepsilon+2$, respectively, and $\varphi-\varepsilon$ of which have size 1, if $\varepsilon \le \varphi$ and $\operatorname{char} R = 0$;

(ii) as φ Jordan blocks of sizes $\varepsilon+\varphi, \varepsilon+\varphi-2, \ldots, \varepsilon-\varphi+2$, respectively, if $\varphi \le \varepsilon$ and $\operatorname{char} R = 0$.

(iii) as φ Jordan blocks, $\min(\varepsilon,\varphi)$ of which have sizes ranging from $\varphi+\varepsilon$ to $\varphi+\varepsilon+2-2\min(\varepsilon,\varphi)$, and $\varphi-\min(\varepsilon,\varphi)$ of which have size 1, if $\operatorname{char} R = p > 0$.

Proof. Lemma 5.5 describes the copies of M in $e_a f_a M/\operatorname{soc} e_a f_a M$ and lemma 5.7 above tells us exactly how $\operatorname{soc}(e_a f_a M) = \varphi M$ fits into the Jordan block structure of c_n. As $c_n \varphi M = 0$ $\operatorname{soc}(c_n e_a f_a M) = \min(\varepsilon,\varphi)M$, these $\min(\varepsilon,\varphi)$ copies of M must lie at the "bottom" of the $\min(\varepsilon,\varphi)$ Jordan blocks described in lemma 5.5. In other words, we get the same description for the action of c_n as for $e_a f_a M/\operatorname{soc} e_a f_a M$ except that each block is size one larger. Furthermore, there are $\varphi - \min(\varepsilon,\varphi)$ singleton Jordan blocks, that is to say Jordan blocks of size 1, as these are the copies that have 0 intersection with $c_n e_a f_a M$ by the same argument as in the proof of lemma 5.7 above.

We warn the reader that although it is tempting to compare the action of c_n on the copies of M in $e_a f_a M$ to the action of $x \otimes 1 + 1 \otimes y$ on $R[x]/(x^{\varepsilon+1}) \boxtimes R[y]/(y^{\varphi})$, this would be incorrect as the case $\operatorname{char} R = p > 0$ shows.

6. $f_a e_a M$ AND c_n

We use the results of the previous section to prove the analogue to Theorem 5.8 for $f_a e_a M$.

Lemma 6.1. *We have maps*

$$f_a e_a M \xrightarrow{\psi} e_a f_a M \longrightarrow e_a f_a M / \operatorname{soc}(e_a f_a M)$$

where the composition is a surjection, and

$$Q \longrightarrow e_a f_a M \xrightarrow{\hat{\psi}} f_a e_a M$$

where the composition is an injection.

Proof. We will define a map $\psi : \operatorname{Ind} e_a M \boxtimes aJ_m \to e_a f_a M$ which will necessarily factor through $\operatorname{pr}_\lambda \operatorname{Ind} e_a M \boxtimes aJ_m = f_a e_a M$ for $m \gg 0$, because $e_a f_a M \in \operatorname{Rep} H_n^\lambda$ and $\operatorname{pr}_\lambda$ takes the largest quotient that is an H_n^λ-module. We will also denote the map $f_a e_a M \to e_a f_a M$ by ψ. Later we shall show ψ is a refinement of the Mackey decomposition map.

Take $m \geq \varepsilon + \varphi$. Define

$$\begin{aligned} \psi : \operatorname{Ind} e_a M \boxtimes aJ_m &\longrightarrow e_a f_a M \qquad \text{by} \\ 1 \otimes (u \otimes y_0) &\stackrel{\psi}{\longmapsto} T_n \otimes (u \otimes v_0) \end{aligned}$$

where $y_j = (X_n - a)^j \in aJ_m$, $v_j = (X_{n+1} - a)^j \in aJ_\varphi$, and $u \in e_a M$ (which we can also think of as sitting in M). (Really we should write $\overline{T_n \otimes (u \otimes v_0)}$ to denote the image of $T_n \otimes (u \otimes v_0) \in \operatorname{Ind} M \boxtimes aJ_\varphi$ in $e_a f_a M$, but we reserve the bar to mean modulo $\operatorname{soc} e_a f_a M$ below.)

By Frobenius reciprocity, it suffices to show ψ is a well-defined $H_{n-1} \otimes H_1$-map on $1 \otimes e_a M \boxtimes aJ_m$. For $h \in H_{n-1}$, $\psi(h \otimes (u \otimes y_0)) = \psi(1 \otimes (hu \otimes y_0)) = T_n \otimes (hu \otimes v_0) = hT_n \otimes (u \otimes v_0)$. By definition, $\psi(1 \otimes (u \otimes y_1)) = \psi((X_n - a) \otimes (u \otimes y_0)) = (X_n - a)T_n \otimes (u \otimes v_0)$, so we need only verify $(X_n - a)^m T_n \otimes (u \otimes v_0) = 0$.

This follows as $m \geq \varepsilon + \varphi$, $(X_{n+1} - a)^\varphi v_0 = 0$, $(X_n - a)^\varepsilon u = 0$, and

$$\begin{aligned} (X_n - a)^m T_n = T_n (X_{n+1} - a)^m + (1 - q) X_{n+1} \big(&(X_n - a)^{m-1} + \\ &(X_n - a)^{m-2}(X_{n+1} - a) + \cdots + (X_{n+1} - a)^{m-1} \big). \end{aligned}$$

Next, we will show the composition

$$\begin{aligned} \operatorname{Ind} e_a M \boxtimes aJ_m &\xrightarrow{\psi} e_a f_a M \longrightarrow e_a f_a M/\operatorname{soc}(e_a f_a M) \qquad (**) \\ 1 \otimes (u \otimes y_0) &\mapsto T_n \otimes (u \otimes v_0) \mapsto \overline{T_n \otimes (u \otimes v_0)} \end{aligned}$$

is a surjection.

By the affine Mackey decomposition and claim 5.2, $\operatorname{Ind} e_a M \boxtimes aJ_\varphi \to e_a f_a M/\operatorname{soc}(e_a f_a M)$ is surjective. The composition of these maps

$$\operatorname{Ind} e_a M \boxtimes aJ_m \twoheadrightarrow \operatorname{Ind} e_a M \boxtimes aJ_\varphi \twoheadrightarrow e_a f_a M/\operatorname{soc}(e_a f_a M)$$

is given by

$$1 \otimes (u \otimes y_0) \mapsto 1 \otimes (u \otimes v_0) \mapsto \overline{T_n \otimes (u \otimes v_0)}$$

and hence coincides with the composition of maps in $(**)$, which is therefore surjective.

For the second statement, recall Q was defined via

$$0 \longrightarrow Q \longrightarrow e_a f_a M \longrightarrow \varphi M \longrightarrow 0.$$

Thus we may use the fact $f_a M = \check{\mathrm{pr}}_\lambda \check{\operatorname{Ind}} M \boxtimes aJ_\varphi$, (where $\check{\mathrm{pr}}_\lambda$ takes the largest submodule that is an H_n^λ-module) and proceed as above, reversing all arrows and denoting the map $e_a f_a M \to f_a e_a M$ by $\hat{\psi}$.

Remark 6.2. *The maps ψ and $\hat{\psi}$ above are refinements of the cyclotomic Mackey decomposition map in the sense that they coincide with the Mackey map composed with the natural inclusion and projection maps*

$$\begin{aligned} f_a e_a M \hookrightarrow rM \bigoplus_a f_a e_a M \xrightarrow{\simeq} \bigoplus_a e_a f_a M \to e_a f_a M \\ e_a f_a M \hookrightarrow \bigoplus_a e_a f_a M \xrightarrow{\simeq} rM \bigoplus_a f_a e_a M \to f_a e_a M. \end{aligned}$$

We note the obvious but useful fact

Claim 6.3. *Let $A, B, C \in \operatorname{Rep} H_n^\lambda$. Given maps $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, if $\beta \circ \alpha$ is a surjection, so is β. If $\beta \circ \alpha$ is a surjection and $\operatorname{cosoc} C \simeq \operatorname{cosoc} B$, then α must also be a surjection.*

Given maps $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$, if $\beta \circ \alpha$ is an injection, so is α. If $\beta \circ \alpha$ is an injection and $\operatorname{soc} A \simeq \operatorname{soc} B$, then β must also be an injection.

Theorem 6.4. *If $\varphi \leq \varepsilon$, the maps ψ and $\hat{\psi}$ of lemma 6.1 are a surjection and injection, respectively:*

$$f_a e_a M \twoheadrightarrow e_a f_a M \quad \textit{and} \quad e_a f_a M \hookrightarrow f_a e_a M.$$

If $\varepsilon \leq \varphi$, then the maps ψ and $\hat{\psi}$ of lemma 6.1 are an injection and surjection, respectively:

$$f_a e_a M \hookrightarrow e_a f_a M \quad \text{and} \quad e_a f_a M \twoheadrightarrow f_a e_a M.$$

Further $f_a e_a M \simeq e_a f_a M/(\varphi - \varepsilon)M$.

Proof. If $\varphi \leq \varepsilon$, from corollary 5.6 and claim 5.2, we see

$$\operatorname{cosoc}(e_a f_a M/\operatorname{soc}(e_a f_a M)) \simeq \min(\varepsilon, \varphi)M \simeq \varphi M \simeq \operatorname{cosoc} e_a f_a M.$$

Because the map $f_a e_a M \to e_a f_a M/\operatorname{soc}(e_a f_a M)$ of lemma 6.1 is a surjection, claim 6.3 implies the map $\psi : f_a e_a M \to e_a f_a M$ is a surjection as well.

Similarly $\operatorname{soc} Q = \min(\varepsilon, \varphi)M = \varphi M = \operatorname{soc} e_a f_a M$, so the map $\hat{\psi}$: $e_a f_a M \to f_a e_a M$ of lemma 6.1 is an injection.

If $\varepsilon \leq \varphi$, consider

$$f_a e_a M \xrightarrow{\psi} e_a f_a M \longrightarrow e_a f_a M/(\varphi - \varepsilon)M \longrightarrow e_a f_a M/\operatorname{soc}(e_a f_a M)$$

where the submodule $(\varphi - \varepsilon)M$ are those $\varphi - \varepsilon$ singleton (size 1) Jordan blocks for c_n. By lemma 5.7, they are the copies of M in both the socle and cosocle of $e_a f_a M$. Thus $\operatorname{cosoc}(e_a f_a M/(\varphi - \varepsilon)M) \simeq \varepsilon M \simeq \operatorname{cosoc}(e_a f_a M/\operatorname{soc}(e_a f_a M)$. By lemma 6.1 the above composition of maps is a surjection, so applying claim 6.3, we have

$$f_a e_a M \longrightarrow e_a f_a M/(\varphi - \varepsilon)M$$

is a surjection.

If that surjection had nontrivial kernel, this kernel would intersect $\operatorname{soc} f_a e_a M \simeq \varepsilon M$. However there are exactly $\varphi\varepsilon + \varepsilon$ copies of M in each module, showing the map is an isomorphism and in particular an injection. Thus ψ is also an injection.

Because $(\varphi - \varepsilon)M$ is both in the socle and cosocle of $e_a f_a M$, it splits off. Denote its direct complement by Q'. Then $\operatorname{soc} Q' = \varepsilon M = \operatorname{soc} Q$. Recall the composition

$$Q \to Q' \to e_a f_a M \xrightarrow{\hat{\psi}} f_a e_a M$$

is an injection. Therefore, by claim 6.3, the map

$$Q' \to f_a e_a M$$

is also an injection. Because each of Q' and $f_a e_a M$ contains $\varphi\varepsilon + \varepsilon$ copies of M, and their cosocles consist only of copies of M, that map must be an isomorphism (inverse to the isomorphism above) and in particular a surjection. Therefore $\hat{\psi}$ is also a surjection.

Although we are in a position to prove Theorem 7.1 now, we will defer it to section 7, and first will apply the results of Theorem 6.4 to prove the analogue to Theorem 5.8 for $f_a e_a M$.

Theorem 6.5. *The action of c_n on the $\varepsilon(\varphi+1)$ copies of M in $f_a e_a M$ is*

(i) as ε Jordan blocks, φ of which have sizes $\varepsilon+\varphi, \varepsilon+\varphi-2, \ldots, \varepsilon-\varphi+2$, respectively, and $\varepsilon-\varphi$ of which have size 1, if $\varphi \le \varepsilon$ and $\mathrm{char} R = 0$;

(ii) as ε Jordan blocks of sizes $\varepsilon+\varphi, \varepsilon+\varphi-2, \ldots, \varepsilon-\varphi+2$, respectively, if $\varepsilon \le \varphi$ and $\mathrm{char} R = 0$.

(iii) as ε Jordan blocks, $\min(\varepsilon,\varphi)$ of which have sizes ranging from $\varepsilon+\varphi$ to $\varepsilon+\varphi+2-2\min(\varepsilon,\varphi)$, and $\varepsilon-\min(\varepsilon,\varphi)$ of which have size 1, if $\mathrm{char} R = p > 0$.

Proof. In the case $\varphi \le \varepsilon$, Theorem 6.4 gives an injection $\hat{\psi} : e_a f_a M \hookrightarrow f_a e_a M$. Because, by claim 5.2, $\mathrm{soc}\, e_a f_a M = \varphi M$ and $\mathrm{soc}\, f_a e_a M = \varepsilon M$, there is a submodule $(\varepsilon-\varphi)M \subseteq f_a e_a M$ that has 0 intersection with $\hat{\psi}(e_a f_a M)$. The image of $e_a f_a M$ accounts for the remaining $\varepsilon\varphi+\varphi$ copies of M in $f_a e_a M$, and the action of c_n on them is described in Theorem 5.8. Thus c_n must act on those $(\varepsilon-\varphi)$ copies of M as singleton Jordan blocks.

In the case $\varepsilon \le \varphi$, by Theorem 6.4, $e_a f_a M$ contains a submodule $Q' \simeq f_a e_a M$. Similar to the above case, Q' has 0 intersection with $(\varphi-\varepsilon)M$. Further, Q' accounts for the remaining $\varepsilon\varphi+\varepsilon$ copies of M in $e_a f_a M$. Thus $(\varphi-\varepsilon)M$ accounts for the singleton Jordan blocks described in Theorem 5.8. The remaining Jordan blocks describe the action of c_n on the copies of M in $f_a e_a M$.

7. $[e_a, f_a] = h_a$

We conclude by noting that the maps ψ and $\hat{\psi}$ are splittings. Depending on the relative sizes of $\varphi_a(M)$ and $\varepsilon_a(M)$, one of $e_a f_a M$ and $f_a e_a M$ is a direct summand of the other.

Theorem 7.1. *If $\varepsilon \le \varphi$ then $e_a f_a M \simeq f_a e_a M \bigoplus (\varphi-\varepsilon)M$.*
If $\varphi \le \varepsilon$ then $f_a e_a M \simeq e_a f_a M \bigoplus (\varepsilon-\varphi)M$.

Proof. In the case $\varepsilon \le \varphi$, by Theorem 6.4 $f_a e_a M \simeq e_a f_a M/(\varphi-\varepsilon)M$. Further $(\varphi-\varepsilon)M$ are the copies of M in both the socle and cosocle of $e_a f_a M$, and as such is a direct summand. In other words, $f_a e_a M \oplus (\varphi-\varepsilon)M \simeq e_a f_a M$.

Suppose $\varphi \le \varepsilon$. By Theorem 6.4, $f_a e_a M$ has a submodule Q which is isomorphic to $e_a f_a M$. Comparing socles, $Q \cap (\varepsilon-\varphi)M = 0$, so that $f_a e_a M$ has a submodule isomorphic to $e_a f_a M \bigoplus (\varepsilon-\varphi)M$.

Because each of these contains $\varepsilon\varphi+\varepsilon$ copies of M and the cosocle of $f_a e_a M$ consists only of copies of M, we must have $e_a f_a M \bigoplus (\varepsilon-\varphi)M \simeq$

$f_a e_a M$. Notice that as above, the direct summand $(\varepsilon - \varphi)M$ accounts for the singleton Jordan blocks for the action of c_n and are also the copies of M in both the socle and cosocle of $f_a e_a M$.

Corollary 7.2. *$e_a f_a M / \operatorname{soc}(e_a f_a M) \simeq f_a e_a M / \operatorname{soc}(f_a e_a M)$; $c_n e_a f_a M \simeq c_n f_a e_a M$.*

One interpretation of Theorem 7.1 is that $[e_a, f_a] = e_a f_a - f_a e_a = \varphi_a - \varepsilon_a$, i.e. as operators on $K(\operatorname{Rep} H_n^\lambda)$

$$[e_a f_a M] - [f_a e_a M] = (\varphi_a(M) - \varepsilon_a(M))[M],$$

where here the brackets denote equivalence classes in the Grothendieck group (but above is a Lie bracket). However, Theorem 7.1 is saying something stronger: essentially that $[e_a, f_a] = e_a f_a - f_a e_a$ (or $[f_a, e_a] = f_a e_a - e_a f_a$) makes sense $\operatorname{Rep} H_n^\lambda$ *before* passing to the Grothendieck group if we interpret subtraction as meaning quotienting out a direct summand.

The strong relationship between e_a and f_a and the importance of the quantity $\varphi_a(M) - \varepsilon_a(M)$ are investigated further and explained in [G].

In particular, with the full version of Theorem 2.4, i.e. the alternate definition (2) of φ_a, the quantity $h_a(M) = \varphi_a(M) - \varepsilon_a(M)$ defines the weight of M which can be shown to depend only on the central character of M.

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