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PARAMETERIZING HECKE ALGEBRA MODULES: BERNSTEIN–ZELEVINSKY MULTISEGMENTS, KLESHCHEV MULTIPARTITIONS, AND CRYSTAL GRAPHS

M. VAZIRANI*

UC Berkeley
Department of Mathematics
970 Evans Hall # 3840
Berkeley, CA 94720-3840, U.S.A.

vazirani@math.berkeley.edu

Abstract. This paper provides a combinatorial and representation-theoretic dictionary between three sets of objects: multisegments, multipartitions, and the irreducible modules of the affine Hecke algebra H_n of type A (for generic q). The dictionary is dictated by Grojnowski's theorem [G] that gives a crystal structure on the irreducible modules of $H_n, n \geqslant 0$. (See also [A2], [AM].) Given a Kleshchev multipartition, we construct an irreducible module as the unique simple quotient of a module induced up from smaller Hecke algebras. We compare that to Bernstein–Zelevinsky's [BZ, Z] construction of an irreducible module given a multisegment. We describe the corresponding crystal structure on multisegments and multipartitions. However, the new result here is not the explicit description of the crystal graph, but the identification of three different constructions of the irreducible modules, all of which give you the *same* graph when the edges are given by taking simple submodules in a restriction.

1. Introduction

This paper gives a construction of the irreducible modules of the affine Hecke algebra of type A, $H_n = H_n^{\mathrm{aff}}(q)$ (for q generic) and discusses related combinatorics. The initial step is to describe the results of Bernstein–Zelevinsky [BZ], [Z], who gave the first complete parameterization of the irreducibles in terms of combinatorial objects called "multisegments." From that combinatorial data, they also gave a direct construction of each irreducible module as a distinguished composition factor of a module induced up from trivial modules of smaller Hecke algebras. (This was actually done in the context of the representation theory of $GL(n, F)$ where F is a non-Archimedean local field, but easily translates to the language of Hecke algebras. See Section 6. See also [Ro].)

We do a similar thing here. The combinatorial objects "Kleshchev multipartitions" parameterize certain irreducible modules, and we use that combinatorial data to construct them as a distinguished composition factor of a module induced up from smaller Hecke algebras. This paper also identifies this parameterization with that of multiseg-

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ments and with a third parameterization by the nodes of a "crystal graph."

In [G], Grojnowski parameterized the irreducible modules of H_n by the nodes of a crystal graph. (See [A2], [AM] where also a crystal structure on the irreducible modules is defined.) In Theorem of [G] (repeated here as Theorems 10.1 and 10.2) he constructs the crystal graph whose nodes correspond to the irreducible modules of H_n and proves that graph is $B(\infty)$ (Kashiwara's crystal graph associated to $U(\mathfrak{n}^-)$, where $\mathfrak{n}^-$ is the nilpotent subalgebra of $\mathfrak{gl}_\infty$). Grojnowski furthermore introduces a representation-theoretic operation corresponding to the edges of the crystal graph, by which one can also construct the irreducible modules. An edge colored by i connects irreducible modules $N \xrightarrow{i} M$ iff N is a simple submodule of the "i-restriction" of M (the unique one, in fact, and then we write $N = \widetilde{e}_i M$). We point out that Grojnowski's and Ariki's results hold even when q is a root of unity, but we only discuss the case of generic q here.

We give the comparison between multisegments and nodes of $B(\infty)$ in Theorem 3.1, where we show [BZ, Z]'s map from multisegments to irreducible modules

$$\mathbf{\Delta} \mapsto M_{\mathbf{\Delta}}$$

intertwines the action of the crystal operator $\widetilde{\mathbf{e}}_\mathbf{i}$ on multisegments Section 2.3) with the action of $\widetilde{e}_i$ on modules. (The notation is explained in Section 2 below.) Rule 1 (or Rule 2) of Section 2.3 gives the action of $\widetilde{\mathbf{e}}_\mathbf{i}$ on a multisegment $\mathbf{\Delta}$. This is not new — Reineke discusses this crystal structure in [Re]. However, what is new is showing that Bernstein–Zelevinsky's representation-theoretic interpretation of the nodes, i.e., the identification of the multisegment $\mathbf{\Delta}$ with the irreducible module $M_{\mathbf{\Delta}}$ coincides with that of Grojnowski.

We remark that the above statement is not obvious. For instance, using different crystal operators $\widehat{\widetilde{e}}_i$ (defined below) one can construct the *same* crystal graph but the node $\mathbf{\Delta}$ will now correspond to a module $\mathrm{rev}^* M_{\mathbf{\Delta}}$, where rev^* is Zelevinsky's involution. See [MW], [Z2].

Another byproduct of Theorem 3.1 below is the determination of which multisegments parameterize modules of the *cyclotomic* Hecke algebra H_n^λ (in Theorem 3.2), where H_n^λ is a certain finite-dimensional quotient of H_n. In the language of crystals, this corresponds to describing how the crystal graph $B(\lambda)$ is a subgraph of $B(\infty)$. Grojnowski has shown in Theorem 14.3 of [G] that the subgraph of $B(\infty)$ whose nodes correspond to the irreducible modules of H_n^λ is $B(\lambda)$, the crystal graph associated to the irreducible highest weight representation of $\mathfrak{gl}_\infty$ with highest weight λ.

Finally, we give a new construction of the irreducible modules of H_n^λ from the combinatorial data of multipartitions, which can also be used to label the nodes of the crystal graph $B(\lambda)$. Again, there is a crystal structure on multipartitions. In fact there are several crystal structures — we have to choose the correct one. Indeed, we only consider the "Kleshchev multipartitions" here (see Section 2.4), which come from identifying $B(\lambda)$ as a subgraph of $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$ when $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_r$. Given a λ-colored Kleshchev multipartition $\underline{\mu}$, we construct an irreducible module $N_{\underline{\mu}[\lambda]}$ as the unique simple quotient of an induced module and prove in Theorem 3.5 that the map from

$$\underline{\mu} \mapsto N_{\underline{\mu}[\lambda]}$$

intertwines the action of $\widetilde{\mathbf{e}}_\mathbf{i}$ on Kleshchev multipartitions (Rule 3) with the action of $\widetilde{e}_i$

on irreducible modules.

If we change the order of the i_k above, then the resulting parameterization of the nodes of $B(\lambda)$ is by *different* multipartitions which do not have as nice a representation-theoretic interpretation; for instance, the analogous construction of irreducible modules fails. Further, the parameterization by Kleshchev multipartitions also appears in the work of [A2, AM, M] although they use a different construction.

We describe our construction of $N_{\underline{\mu}[\lambda]}$ as direct, meaning that the module $N_{\underline{\mu}[\lambda]}$ is given as the unique simple quotient of an induced module (similar to [B, BZ]). Grojnowski's result gives a construction by first choosing a path on the crystal graph $B(\lambda)$ from the highest weight node to the node labeled $\underline{\mu}$, and then following it, inducing and taking quotients of modules n times to construct an H_n-module. And so we consider this n-step construction indirect (however powerful) compared to our one-step construction in Theorem 3.4. We only have to compute a simple quotient once. It is not true in general that taking quotients commutes with inducing, and this is the representation-theoretic difficulty we overcome.

The proofs given here do not rely on [G] Theorem 14.3, but do use several results from [G], [GV] (and of course from [BZ] and [Z]). One reason for this is that the proofs are relatively straightforward. Another is that several steps in the proofs given here are necessary ingredients for proving Theorems 3.4 and 3.5.

There are related results in the literature in [A2, AM, G2]. Both our proofs and construction are different from theirs, and we comment on this at appropriate points in the text.

This introduction and Section 10 translate the main theorems of this paper into the language of crystal graphs. This is worthwhile, as the theorems below partially explain why the rule for computing $\widetilde{e}_i$ mirrors the rule we know for that on a tensor product of crystal graphs. However, the rest of the sections avoid that language and simply deal with algebras and modules. No background in crystal graphs is necessary to understand the proofs (but it is helpful to appreciate them).

We begin with some necessary definitions and notation, then immediately state the main results. However, the reader may want to read the more extensive definitions is Section 4 before section 3. In Sections 8 and 9 the proofs are given. In Section 6 some results and proofs that appear in [BZ, Z] are given, recast in this notation, and included for convenience to the reader.

We remind the reader that these proofs require q to be generic, whereas the theorems of [G] hold for all $q \in R^{\times}$. We expect some of the results of this paper to generalize to the case when q is a root of unity, as evidenced by Grojnowski's results as well as the work of [Br], [Kv1], [Kv2], [Kv3], [LLT] pertaining to the modular representation theory of S_n or of $H_n^{\Lambda_0}$ at a root of unity.

2. Background notation and results

In the following subsections, we define three sets of objects which will be the nodes of three different crystal graphs. These are the irreducible modules of H_n in the subcategory Rep_q, multisegments, and colored multipartitions. We define the action of an operator $\widetilde{e}_i$ on each of these three sets, i.e., we describe the edges of the crystal graphs. In Section 3 are the theorems showing the three crystal graphs are isomorphic.

2.1. H_n^{fin}, H_n and H_n^λ

Throughout the paper we fix an algebraically closed field R, and an invertible element $q \in R$ such that $q^\ell = 1$ implies $\ell = 0$. In this situation $\{q^i\}_{i \in \mathbb{Z}}$ is infinite and we say q is *generic*.

The *finite Hecke algebra*, H_n^{fin} is the R-algebra with generators

$$T_1, \dots, T_{n-1}$$

and following relations:

$$\begin{aligned} &\textit{braid relations} && T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i,\ |i-j| > 1, && (1)\\ &\textit{quadratic relations} && (T_i+1)(T_i-q) = 0. && (2) \end{aligned}$$

The braid relations imply that if $w = s_{i_1} \cdots s_{i_k}$ and $\ell(w) = k$, then $T_{i_1} \cdots T_{i_k}$ depends only on $w \in S_n$. It is denoted T_w, and the $\{T_w \mid w \in S_n\}$ form a basis of H_n^{fin} over R.

The *affine Hecke algebra* [BZ] H_n (or H_n^{aff}) is the R-algebra, which as an R-module is isomorphic to

$$H_n^{\mathrm{fin}} \otimes_R R[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$

The algebra structure is given by requiring that H_n^{fin} and $R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be subalgebras, and that

$$T_iX_iT_i = qX_{i+1}. \tag{3}$$

Denote by $\operatorname{Rep}_q H_n$ the category of finite-dimensional modules M for H_n such that the only eigenvalues of the X_j on M are powers of q. Let $\operatorname{Rep}_q = \bigoplus_{n \geqslant 0} \operatorname{Rep}_q H_n$.

Remark 1. It follows from the computation in [G], Section 6.2, or is explained in [V], or in many other places, that to understand $\operatorname{Rep} H_n$ it is enough to understand $\operatorname{Rep}_q H_n$.

The *cyclotomic Hecke algebra* or *Ariki–Koike algebra* [AK] H_n^λ is the quotient H_n/I_λ, where I_λ is the ideal generated by the polynomial in X_1: $\prod (X_1 - q^i)^{m_i}$, and $\lambda = \sum m_i \Lambda_i$ is a weight of $\mathfrak{gl}_\infty$ (where $\{\Lambda_i \mid i \in \mathbb{Z}\}$ denote the fundamental weights). We may also write $\lambda = \sum_{k=1}^r \Lambda_{i_k}$ and so I_λ is generated by $\prod_{k=1}^r (X_1 - q^{i_k})$.

Remark 2. Observe that any irreducible module in Rep_q is an irreducible H_n^λ-module if we take λ large enough. Just let $\prod_{i=1}^r (X_1 - q^i)^{m_i}$ be the characteristic polynomial of X_1 acting on the module. Conversely, we identify any irreducible H_n^λ-module as an irreducible H_n-module (on which $\prod_{i=1}^r (X_1 - q^i)^{m_i}$ vanishes).

2.2. e_i, $\widetilde{e}_i$, and ε_i

Given an irreducible H_n-module M, consider $\operatorname{Res}_{n-1,1}^n M = \operatorname{Res}_{H_{n-1} \otimes H_1}^{H_n} M$. As $X_n - q^i \in Z(H_{n-1} \otimes H_1)$, left multiplication by $(X_n - q^i)^m$ induces an endomorphism of $\operatorname{Res}_{H_{n-1} \otimes H_1}^{H_n} M$, and its kernel for $m \gg 0$ (we can take $m \geqslant \dim M$) is the generalized eigenspace of $X_n - q^i$. Define e_iM to be the restriction of that generalized eigenspace to H_{n-1}. The functor e_i, or i-restriction, appears in the work of Robinson, as well as that of [A], [G], [LLT], and others. Because restriction and taking generalized eigenspaces are exact functors, we have the following claim.

Claim 1. *e_i is an exact functor* $\operatorname{Rep}_q H_n \to \operatorname{Rep}_q H_{n-1}$ *and* $\operatorname{Res}^{H_n}_{H_{n-1}} M = \bigoplus_{i\in\mathbb{Z}} e_i M$.

Define

$$\varepsilon_i(M) = \max\{m \geqslant 0 \mid e_i^m M \neq 0\}.$$

We also define

$$\widetilde{e}_i M = \operatorname{soc} e_i M.$$

Observe $\varepsilon_i(M) = \max\{m \geqslant 0 \mid \widetilde{e}_i^m M \neq 0\}$ as well. We remind the reader that the socle of a module N, denoted $\operatorname{soc}(N)$, is the largest semisimple submodule of N, and that the cosocle of N, denoted $\operatorname{cosoc}(N)$, is its largest semisimple quotient.

We recall from [GV], Theorem B.

Theorem 2.1. ([GV], Theorem B) *Let M be an irreducible H_n-module. Then $\widetilde{e}_i M$ is zero or is irreducible, and* $\operatorname{cosoc} e_i M$ *is isomorphic to $\widetilde{e}_i M$.*

Similarly, given $M \in \operatorname{Rep}_q H_n$, define $\widehat{e}_i M$ to be the $X_1 - q^i$ generalized eigenspace of $\operatorname{Res}^n_{1,n-1} M$, further restricted to H_{n-1}. (Technically, one should then reindex the T_k and X_k to T_{k-1} and X_{k-1}.) If M is irreducible, let $\widehat{\widetilde{e}}_i M = \operatorname{soc} \widehat{e}_i M$ and $\widehat{\varepsilon}_i(M) = \max\{m \geqslant 0 \mid \widehat{e}_i^m M \neq 0\}$. Analogous to Theorem 2.1, if M is irreducible, then $\widehat{\widetilde{e}}_i M$ is irreducible or zero.

2.3. Multisegments

Write $\triangle_{(i,j)} = (q^i q^{i+1} \cdots q^j)$, $i \leqslant j$ for the one-dimensional trivial representation of H_{j-i+1} on which each $T_k - q$ and $X_k - q^{k+i-1}$ vanishes. We refer to $\triangle_{(i,j)}$ as a *segment*. Adopt the convention that $\triangle_{(j,j-1)} = \mathbf{1}$, the one dimensional H_0-module, or that $\{\triangle_{(j,j-1)}\} = \varnothing$.

We introduce the symbol $\mathbf{0}$, not to be confused with $\varnothing$, which will stand for the zero module.

Call a multiset of segments a *multisegment*. Section 6 outlines many of the results and proofs of [BZ, Z] regarding multisegments, but given in the language of Hecke algebras. Theorem 2.2 below is the result from [BZ, Z] that multisegments classify the irreducible modules of Rep_q.

In the following theorems, we will consider two total orderings on segments. Examples will be given in Section 7.

Right order: Order segments so that $\triangle_{(i_1,j_1)} > \triangle_{(i_2,j_2)}$ if $i_1 > i_2$ or if $i_1 = i_2$ and $j_2 > j_1$.

Left order: Order segments so that $\triangle_{(i_1,j_1)} \succ \triangle_{(i_2,j_2)}$ if $j_1 > j_2$ or if $j_1 = j_2$ and $i_2 > i_1$.

Here we define a function from multisegments to multisegments $\cup\{\mathbf{0}\}$

$$\boldsymbol{\Delta} \mapsto \widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta}.$$

Rule 1. Given a multisegment $\boldsymbol{\Delta}$, first put $\boldsymbol{\Delta}$ in the right order, so that $\boldsymbol{\Delta} = \{\triangle_{(a_1,b_1)}, \triangle_{(a_2,b_2)}, \ldots, \triangle_{(a_m,b_m)}\}$ with $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_m$. We write its "$j$-signature" as follows. Assign a blank to any segment such that $b_k \neq j, j-1$, assign $-$ if $b_k = j$, and $+$ if $b_k = j-1$. In the corresponding word $\pm\pm\cdots\pm$ (interspersed with blanks), cancel any adjacent $-+$, continuing, ignoring all previously cancelled symbols until what is left uncanceled has the form

$$++\cdots+--\cdots-.$$

Suppose that the leftmost uncanceled $-$ (if it exists) is in position i, i.e., that it came from $\triangle_{(a_i,j)}$. Define

$$\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta} = \boldsymbol{\Delta} \cup \{\triangle_{(a_i,j-1)}\} \setminus \{\triangle_{(a_i,j)}\},$$

and $\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta} = \mathbf{0}$ if no such $-$ exists.

Observe that the $\pm$ word corresponding to $\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta}$, if $\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta} \neq \mathbf{0}$, is exactly that of $\boldsymbol{\Delta}$ except that the leftmost uncanceled $-$ has been changed to $+$. Further, if ε is the total number of uncanceled $-$ signs, then $\widetilde{\mathbf{e}}_{\mathbf{j}}^{\varepsilon+1}\boldsymbol{\Delta} = \mathbf{0}$.

An example of computing $\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta}$ is given in Section 7.

Define a second function from multisegments to multisegments $\cup\{\mathbf{0}\}$

$$\boldsymbol{\Delta} \mapsto \widehat{\widetilde{\mathbf{e}}}_{\mathbf{i}}\boldsymbol{\Delta}.$$

Rule 2. Given a multisegment $\boldsymbol{\Delta}$, we define $\widehat{\widetilde{\mathbf{e}}}_{\mathbf{i}}\boldsymbol{\Delta}$ as follows: put $\boldsymbol{\Delta}$ into the left order. For each $\triangle_{(i,z)}$ write $-$, for each $\triangle_{(i+1,z')}$ write $+$, and for all other segments write a blank. Now we ignore all $+-$ pairs, similar to Rule 1, which will leave uncanceled symbols $-\cdots-+\cdots+$. Then, if $\triangle_{(i,z)}$ corresponds to the rightmost $-$, we replace that segment with $\triangle_{(i+1,z)}$. If no such uncanceled $-$ exists, then $\widehat{\widetilde{\mathbf{e}}}_{\mathbf{i}}\boldsymbol{\Delta} = \mathbf{0}$.

This rule agrees with that of [Re].

Given a multisegment and a choice of ordering of its segments $\boldsymbol{\Delta} = \{\triangle_{(i_1,j_1)},\dots$ $\dots,\triangle_{(i_m,j_m)}\}$ let

$$\mathrm{Ind}\boldsymbol{\Delta}$$

denote $\mathrm{Ind}^n_{n_1,n_2,\dots,n_m}\triangle_{(i_1,j_1)}\boxtimes\cdots\boxtimes\triangle_{(i_m,j_m)}$, where $n_k = j_k - i_k + 1$ and $n = \sum_k n_k$, and where $\mathrm{Ind}^B_A M = B\otimes_A M$ (which is discussed further in Section 4.1). Define $n(\boldsymbol{\Delta}) = n$ and $m(\boldsymbol{\Delta}) = m$. (Observe $n(\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta}) = n(\boldsymbol{\Delta}) - 1$ if $\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta} \neq \mathbf{0}$.)

Multisegments are by definition unordered. However, it is necessary when discussing induced modules to have an order understood. From now on, unless stated otherwise, given a multisegment $\boldsymbol{\Delta}$, let $\boldsymbol{\Delta}$ denote the multisegment in the right order, and $\underline{\boldsymbol{\Delta}}$ denote it in the left order.

Define

$$M_{\boldsymbol{\Delta}} = \operatorname{cosoc}\mathrm{Ind}\boldsymbol{\Delta}.$$

Set $M_{\mathbf{0}} = 0$. Again, the symbol $\mathbf{0}$ is not to be confused with the empty segment $\varnothing$ for which $M_\varnothing = \mathbf{1}$, the one dimensional module of $H_0 = R$.

The following theorem of [BZ, Z] show that multisegments parameterize the irreducible modules in $\mathrm{Rep}_q H_n^{\mathrm{aff}}$, $n \geqslant 0$. A proof will be given in Section 6.2.

Theorem 2.2. ([BZ, Z]) *Let $\boldsymbol{\Delta}$ be a multisegment with $n = n(\boldsymbol{\Delta})$. Then*

(1) $M_{\boldsymbol{\Delta}} := \operatorname{cosoc}\mathrm{Ind}\boldsymbol{\Delta}$ *is an irreducible H_n-module.*
(2) *If $\boldsymbol{\Delta} \neq \boldsymbol{\Delta}'$, then $M_{\boldsymbol{\Delta}} \not\cong M_{\boldsymbol{\Delta}'}$.*
(3) *Given any irreducible $M \in \mathrm{Rep}_q H_n$, there exists a multisegment $\boldsymbol{\Delta}$ (with $n(\boldsymbol{\Delta}) = n$) such that $M = M_{\boldsymbol{\Delta}}$.*

The main theorem of Section 3, proved in Section 8, is that $\widetilde{e}_j M_{\boldsymbol{\Delta}} = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta}}$ (and $\widehat{\widetilde{e}}_i M_{\boldsymbol{\Delta}} = M_{\widehat{\widetilde{\mathbf{e}}}_{\mathbf{i}}\boldsymbol{\Delta}}$).

2.4. Partitions

Recall that μ is a partition of n with length k if $\mu = (\mu_1, \ldots, \mu_k)$ such that $\sum_m \mu_m = n$ and $\mu_1 \geqslant \ldots \geqslant \mu_k > 0$. We write $|\mu| = n$.

Definition 1. For a partition μ of length k and for $i \in \mathbb{Z}$, let $\mathbf{\Delta}(\mu, i) = \{\triangle_{(i,i+\mu_1-1)}, \ldots \ldots, \triangle_{(i-k+1,i-k+\mu_k)}\}$. Observe $|\mu| = n(\mathbf{\Delta})$.

We will say μ is colored by i, written $\mu[i]$ if we associate it with $\mathbf{\Delta}(\mu, i)$. We picture it as follows. Consider the Young diagram associated to μ, which consists of μ_1 boxes in the first row, μ_2 boxes in the second, etc. Fill the (x, y) box with $i + x - y$.

$$\begin{array}{|c|c|c|c|c|c|}
\hline
i & i{+}1 & \cdots & \cdots & \cdots & b_1 \\
\hline
i{-}1 & i & \cdots & b_2 \\
\cline{1-4}
\vdots & \vdots & \vdots \\
\cline{1-3}
i{-}k{+}1 & \cdots & b_k \\
\cline{1-3}
\end{array}$$

Then the row fillings correspond to the segments of $\mathbf{\Delta}(\mu, i)$, with $b_m = i + \mu_m - m$. Then we say $\mu[i]$ has a *removable* j-box if you can remove a j-filled box $\boxed{\text{j}}$ from the diagram such that the result is again the diagram of a partition (colored by i). We say $\mu[i]$ has an *addable* j-box if you can add a j-filled box to the diagram such that the result is again the diagram of a partition (colored by i).

We will call an r-tuple of partitions $\underline{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$ a multipartition of n if $n = \sum_m |\mu^{(m)}|$. Likewise, given a multipartition $\underline{\mu}$ and a weight $\lambda = \Lambda_{i_1} + \ldots + \Lambda_{i_r}$ of level r, with $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_r$, we will say $\underline{\mu}$ is colored by λ, written $\underline{\mu}[\lambda]$, if $\mu^{(m)}$ is colored by i_m.

If $\underline{\mu}[\lambda]$ satisfy the condition

$$\mu^{(t)}_{i_t - i_{t+1} + x} \leqslant \mu^{(t+1)}_x \qquad \text{for all } x \geqslant 1,\ 1 \leqslant t \leqslant r-1,$$

then we will say it is a Kleshchev multipartition. The only multipartitions we will consider in Theorems 3.4 and 3.5 will be Kleshchev multipartitions. See Corollary 3.7 for a crystal-theoretic interpretation of Kleshchev multipartitions.

Definition 2. If $\underline{\mu}$ is colored by λ, set $\mathbf{\Delta}(\underline{\mu}[\lambda]) = \bigcup_{j=1}^{r} \mathbf{\Delta}(\mu^{(j)}, i_j)$.

Rule 3. Given a multipartition $\underline{\mu}$ and weight $\lambda = \Lambda_{i_1} + \ldots + \Lambda_{i_r}$ with $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_r$, we define $\widetilde{\mathbf{e}}_{\mathbf{j}}(\underline{\mu}[\lambda])$ as follows. If $\mu^{(k)}$ has a removable j box when colored by i_k, write the symbol $-$. If $\mu^{(k)}$ has an addable j box when colored by i_k, write $+$. Otherwise, write a blank. In the resulting word of length r, cancel all occurrences of $-+$. In the remaining uncanceled symbols $++\cdots+-\cdots-$, we remove the j box from the $\mu^{(k)}$ that corresponds to the leftmost $-$ symbol, if one exists. The resulting colored multipartition is $\widetilde{\mathbf{e}}_{\mathbf{j}}(\underline{\mu}[\lambda])$. Otherwise, $\widetilde{\mathbf{e}}_{\mathbf{j}}(\underline{\mu}[\lambda]) = \mathbf{0}$.

Again, we note that $\widetilde{\mathbf{e}}_{\mathbf{j}}$ "changes" the leftmost $-$ to a $+$.

Example 1. Let $\lambda = \Lambda_4 + 3\Lambda_3 + \Lambda_0$, $\underline{\mu} = ((3), \varnothing, (3,3), (3,3,1,1), (3,1))$. Then $\underline{\mu}[\lambda]$ is a Kleshchev multipartition.

$$\underline{\mu} = \left(\begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline \end{array}, \quad \varnothing, \quad \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 & 3 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 & 3 & 4 \\ \hline \mathbf{1} \\ \cline{1-1} 0 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline -1 \\ \cline{1-1} \end{array} \right)$$

Further, $\widetilde{\mathbf{e}}_{\mathbf{-1}}\underline{\mu} = ((3), \varnothing, (3,3), (3,3,1,1), (3))$ and $\widetilde{\mathbf{e}}_{\mathbf{0}}\underline{\mu} = \mathbf{0}$. However, for $\nu = \big((3), \varnothing, (3,3), (3,2,1,1), (3)\big)$, $\underline{\nu}[\lambda]$ is not a Kleshchev multipartition, because the second row of $\nu^{(3)}$ would be longer than the row of $\nu^{(4)}$ which begins with the same filling.

Definition 3. Define $N_{\mu[i]} = M_{\mathbf{\Delta}(\mu,i)}$. Let the symbol $\mathbf{0}$ be such that $\mathbf{\Delta}(\mathbf{0}, i) = \mathbf{0}$ and $N_{\mathbf{0}[i]} = 0$ (not to be confused with the empty partition $\mu = \varnothing$, in which case $N_{\varnothing[i]} = \mathbf{1}$). Define $N_{\underline{\mu}[\lambda]} = \operatorname{cosoc} \operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$.

In Theorem 3.4, we will see the modules $N_{\underline{\mu}[\lambda]}$ are irreducible when $\underline{\mu}[\lambda]$ is a Kleshchev multipartition. In Section 3, we will also see the rule for $\widetilde{\mathbf{e}}_{\mathbf{j}}$ on colored multipartition is compatible with that for the corresponding multisegment and with that of $\widetilde{e}_j$ on the corresponding irreducible module.

3. Main results

Below are the theorems that give the action of $\widetilde{e}_j$ on irreducible modules of Rep_q both in terms of parameterization by multisegments and by colored multipartitions. The action of $\widetilde{e}_j$ on irreducibles corresponds to the action of $\widetilde{\mathbf{e}}_{\mathbf{j}}$ on multisegments and on Kleshchev multipartitions. In other words, of the three crystal graphs

$$B_{BZ}, \ \ B_{\text{aff}}, \ \ B'_\lambda$$

with edges described by

$$\widetilde{\mathbf{e}}_{\mathbf{i}}\mathbf{\Delta} \xrightarrow{i} \mathbf{\Delta}, \ \ \widetilde{e}_i M \xrightarrow{i} M, \ \ \widetilde{\mathbf{e}}_{\mathbf{i}}(\underline{\mu}[\lambda]) \xrightarrow{i} \underline{\mu}[\lambda]$$

the first two are isomorphic with isomorphism given by

$$\mathbf{\Delta} \mapsto M_{\mathbf{\Delta}}.$$

The third is a connected component of the graph of *all* λ-colored multipartitions, and is isomorphic to the subgraph B_λ of B_{aff} corresponding to $\operatorname{Rep} H_n^\lambda$ for $n \geqslant 0$. That embedding is

$$\underline{\mu}[\lambda] \mapsto N_{\underline{\mu}[\lambda]} \quad \text{ or } \quad \underline{\mu}[\lambda] \mapsto \mathbf{\Delta}(\underline{\mu}[\lambda]).$$

Proofs are postponed until Sections 8 and 9.

Theorem 3.1. *Let $\mathbf{\Delta}$ be any multisegment. Then $\widetilde{e}_j M_{\mathbf{\Delta}} = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}}$.*

We have the analogous theorem below.

Theorem 3.2. *Let $\mathbf{\Delta}$ be any multisegment. Then* $\widehat{e}_i M_{\mathbf{\Delta}} = M_{\widehat{\mathbf{e}}_{\mathbf{i}}\mathbf{\Delta}}$.

As a corollary, we determine which $M_{\mathbf{\Delta}}$ are in $\operatorname{Rep} H_n^\lambda$.

Corollary 3.3. *Let $\mathbf{\Delta}$ be a multisegment with $n = n(\mathbf{\Delta})$. Let $\lambda = \sum m_i \Lambda_i$. Then $M_{\mathbf{\Delta}} \in \operatorname{Rep} H_n^\lambda \iff$ for all i, in the $\pm$ word computed as in Rule 2 to calculate $\widehat{\mathbf{e}}_{\mathbf{i}}\mathbf{\Delta}$, there are $\leqslant m_i$ uncanceled $-$ signs. In other words, $\widehat{\mathbf{e}}_{\mathbf{i}}^{m_i+1}\mathbf{\Delta} = \mathbf{0}$.*

Proof. This follows from Theorem 3.2 and also relies on Theorem 9.13 of [G]: that $\widehat{\varepsilon}_i(M)$ is the maximal size of the Jordan block with eigenvalue q^i for X_1 acting on an irreducible module M. $\square$

Theorem 3.2 describes another crystal structure on B_{BZ}, which we will denote $\widehat{B}_{BZ}$. As colored oriented graphs, B_{BZ} and $\widehat{B}_{BZ}$ are identical. However, the module-theoretic interpretation of their nodes is quite different. If a node $\mathbf{\Delta}$ of B_{BZ} corresponds to an irreducible module $M = M_{\mathbf{\Delta}}$, then the same node $\mathbf{\Delta}$ of $\widehat{B}_{BZ}$ corresponds to the module $\operatorname{rev}^* M_{\mathbf{\Delta}}$ where $\operatorname{rev} : H_n \to H_n$ is the algebra automorphism

$$\operatorname{rev}(X_k) = X_{n+1-k}, \qquad \operatorname{rev}(T_k) = -(T_{n-k} + 1 - q).$$

In fact, it is this involution we really use to deduce Theorem 3.2 from Theorem 3.1. Because rev exchanges X_1 and X_n, any statement made regarding $\widetilde{e}_i$ and an irreducible module M can be made for $\widehat{e}_i$ and $\operatorname{rev}^* M$. See [Z2], [MW] for more about this involution, Zelevinsky's involution, and how to combinatorially describe Γ such that $\operatorname{rev}^* M_{\mathbf{\Delta}} = M_\Gamma$.

Theorem 3.4. *Let $M_{\mathbf{\Delta}}$ be an H_n^λ-module where $\lambda = \Lambda_{i_1} + \Lambda_{i_2} + \ldots + \Lambda_{i_r}$, $i_1 \geqslant \ldots \geqslant i_r$. Then there exists an r-tuple of partitions $\underline{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$ satisfying*

$$\mu^{(t)}_{i_t - i_{t+1} + x} \leqslant \mu^{(t+1)}_x \quad \textit{for all } x \geqslant 1,\ 1 \leqslant t \leqslant r-1 \tag{4}$$

such that $M_{\mathbf{\Delta}} = N_{\underline{\mu}[\lambda]}$ and $\mathbf{\Delta} = \mathbf{\Delta}(\underline{\mu}[\lambda])$.

Conversely, if $\underline{\mu}[\lambda]$ satisfies (4), then $N_{\underline{\mu}[\lambda]} = \operatorname{cosoc} \operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$ is irreducible and isomorphic to $M_{\mathbf{\Delta}}$ for $\mathbf{\Delta} = \mathbf{\Delta}(\underline{\mu}[\lambda])$.

Recall that we call an r-tuple of partitions satisfying condition (4) a Kleshchev multipartition.

Theorem 3.5. *Let $\underline{\mu}[\lambda]$ be a colored multipartition satisfying (4). Then $\widetilde{e}_j N_{\underline{\mu}[\lambda]} = N_{\widetilde{\mathbf{e}}_{\mathbf{j}}(\underline{\mu}[\lambda])}$.*

Further, $\widetilde{\mathbf{e}}_{\mathbf{j}}(\mathbf{\Delta}(\underline{\mu}[\lambda])) = \mathbf{\Delta}(\widetilde{\mathbf{e}}_{\mathbf{j}}(\underline{\mu}[\lambda]))$.

Corollary 3.6. *For fixed λ, and for $\underline{\mu}$ satisfying (4), the modules*

$$N_{\underline{\mu}[\lambda]} = \operatorname{cosoc} \operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$$

are irreducible, distinct, and range over all irreducible H_n^λ-modules. In other words, Kleshchev multipartitions parameterize the irreducible H_n^λ-modules.

It is instructive to compare this result and the following result with similar results of [A, AM, M]. Both their construction of the modules $N_{\underline{\mu}[\lambda]}$ and the proof that they are nonzero are different from ours.

In Section 10, we explain the following corollary, which interprets the preceding theorems in terms of the crystals $B(\lambda)$, $B(\infty)$ and $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$.

Corollary 3.7. *The crystal graph B'_λ is isomorphic to B_λ via*

$$\underline{\mu}[\lambda] \mapsto N_{\underline{\mu}[\lambda]}$$

and is isomorphic to the connected component of the unique node of weight λ in $B(\Lambda_{i_r})\otimes \cdots \otimes B(\Lambda_{i_1})$ via

$$\underline{\mu}[\lambda] \mapsto \mu^{(r)} \otimes \cdots \otimes \mu^{(1)}.$$

4. Further definitions

We need a few more definitions to proceed. Here they are. If A is an R-algebra, we write $\operatorname{Rep} A$ for the category of left A-modules which are *finite-dimensional* as R-modules. We recall that the socle of a module M, denoted $\operatorname{soc}(M)$, is the largest semisimple submodule of M, and that the cosocle of M, denoted $\operatorname{cosoc}(M)$, is its largest semisimple quotient. We also write $Z(A)$ for the center of the algebra A.

If A and A' are two R-algebras with modules M and M' respectively, let $M \boxtimes M'$ denote the $A \otimes A'$-module which is isomorphic to $M \otimes_R M'$ as an R-module and has $A \otimes A'$ action given by

$$(a \otimes a') \cdot (m \otimes m') = am \otimes a'm'.$$

Because R is algebraically closed, if M is an irreducible A-module and M' is an irreducible A'-module, then $M \boxtimes M'$ will be an irreducible representation of $A \otimes A'$, and all such are of this form.

4.1. Induction and restriction

Recall that if $A \subset B$ are R-algebras such that B is finitely generated both as a left and right A-module, the exact functor of restriction

$$\operatorname{Res}^B_A : \operatorname{Rep} B \to \operatorname{Rep} A$$

has left and right adjoints, Ind and $\widehat{\operatorname{Ind}}$ defined by

$$\operatorname{Ind}^B_A : \operatorname{Rep} A \to \operatorname{Rep} B, \qquad M \mapsto B \otimes_A M,$$

$$\widehat{\operatorname{Ind}}^B_A : \operatorname{Rep} A \to \operatorname{Rep} B, \qquad M \mapsto \operatorname{Hom}_A(B, M);$$

i.e.,

$$\operatorname{Hom}_B(\operatorname{Ind} M, N) = \operatorname{Hom}_A(M, \operatorname{Res} N),$$

$$\operatorname{Hom}_B(N, \widehat{\operatorname{Ind}} M) = \operatorname{Hom}_A(\operatorname{Res} N, M).$$

If B is a free A-module, then Ind and $\widehat{\operatorname{Ind}}$ are exact functors also. Further, if $A \subset B \subset C$ are inclusions of R-algebras, we have transitivity of induction and restriction:

$$\operatorname{Res}^B_A \operatorname{Res}^C_B = \operatorname{Res}^C_A, \quad \operatorname{Ind}^C_B \operatorname{Ind}^B_A = \operatorname{Ind}^C_A, \quad \widehat{\operatorname{Ind}}^C_B \widehat{\operatorname{Ind}}^B_A = \widehat{\operatorname{Ind}}^C_A.$$

Now apply these remarks to the affine Hecke algebra. Given a sequence $P = (a_1, \ldots, a_k)$ of non-negative integers summing to n, write $H_P = H_{a_1} \otimes \cdots \otimes H_{a_k}$. We have an obvious embedding

$$H_P = H_{a_1} \otimes \cdots \otimes H_{a_k} \hookrightarrow H_n$$

which makes H_n a free H_P-module. Applying the previous remarks we get exact functors Res, Ind, $\widehat{\operatorname{Ind}}$. When unambiguous from context, we just write Res^n_P or Res for $\operatorname{Res}^{H_n}_{H_P}$, and similarly for Ind to lighten notation. These functors depend on the order $(a_1, \ldots, a_k)$ and not just on the underlying set.

4.2. Characters

Write $S = R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and let $M \in \mathrm{Rep}_q H_n$. Define the generalized S-eigenspace

$$M[\gamma] = \{v \in M \mid (X_i - \gamma_i)^m v = 0,\, m \gg 0,\ \text{for all } 1 \leqslant i \leqslant n\}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n) \in R^n$. Since the X_i commute, we can decompose M into a direct sum of generalized eigenspaces

$$\mathrm{Res}_S^{H_n} M = \bigoplus\nolimits_{\gamma \in \{q^i\}^n} M[\gamma].$$

We define the character of M to be the formal sum

$$\mathrm{ch}\, M = \sum\nolimits_{\gamma \in (R^\times)^n} (\dim M[\gamma])\gamma.$$

Since $S = R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] = R[X_1^{\pm 1}] \otimes_R \cdots \otimes_R R[X_n^{\pm 1}] = H_{(1,1,\ldots,1)}$, we will also write $\gamma = \gamma_1 \boxtimes \gamma_2 \boxtimes \cdots \boxtimes \gamma_n$ for the 1-dimensional S-module with character $\gamma = (\gamma_1, \ldots, \gamma_n)$. For ease, we will also write $\gamma = \gamma_1 \gamma_2 \cdots \gamma_n$ for a term in the character of an H_n-module.

5. Useful propositions

In this section, we collect useful results that we will need later. The results in this section are well known and most can be found in [B], [G], [GV], [Ka], [V].

Proposition 1. *If M is an irreducible H_n-module, then M is finite-dimensional with* $\dim_R M \leqslant n!$.

Consequently, $\mathrm{Rep}\, H_n$ includes all irreducible H_n-modules.

Theorem 5.1 (Kato's Theorem). *Let $q^i K_n$ denote $\mathrm{Ind}^n_{1,1,\ldots,1} q^i \boxtimes \cdots \boxtimes q^i$. Then $q^i K_n$ is irreducible, and is the unique H_n-irreducible module with $q^i q^i \cdots q^i$ occurring in its character.*

From [A] or from [G] Proposition 12.1 we have:

Proposition 2. (Serre relations) *As operators on the Grothendieck group $\bigoplus_n K(\mathrm{Rep}_q H_n)$,*

1. *if $|i-j| > 1$, then $e_i e_j = e_j e_i$.*
2. $e_i^2 e_{i\pm 1} + e_{i\pm 1} e_i^2 = 2 e_i e_{i\pm 1} e_i$.

Remark 3. We can reinterpret Proposition 2, as, first, if $q^j \neq q^{i\pm 1}$, then for any finite-dimensional module M, if $\gamma_1\gamma_2 \cdots q^i q^j \cdots \gamma_n$ occurs in $\mathrm{ch}\, M$, then so does $\gamma_1 \gamma_2 \cdots q^j q^i \cdots \cdots \gamma_n$. (Specifically: $\dim M[(\gamma_1, \ldots, q^i, q^j, \ldots, \gamma_n)] = \dim M[(\gamma_1, \ldots, q^j, q^i, \ldots, \gamma_n)]$.)

Secondly, the string $\gamma_1 \cdots q^i q^{i\pm 1} q^i \cdots \gamma_n$ occurs in $\mathrm{ch}\, M$ if and only if $\gamma_1 \cdots q^i q^i q^{i\pm 1} \cdots \cdots \gamma_n$ *or* $\gamma_1 \cdots q^{i\pm 1} q^i q^i \cdots \gamma_n$ occurs. (Furthermore, the dimensions of the generalized S-eigenspaces spaces satisfy that twice the first is the sum of the next two.)

Kato's Theorem implies that if $\gamma_1 \cdots \underbrace{q^i q^i \cdots q^i}_{k} \cdots \gamma_n$ occurs in $\mathrm{ch}\, M$, it does so with multiplicity at least $k!$.

Proposition 3. ([GV], Lemma 3.5, Part 3) *For any irreducible module* N

$$M = \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_k$$

is irreducible with $\varepsilon_i(M) = \varepsilon_i(N) + k$*, and all other composition factors of* $\operatorname{Ind} N \boxtimes q^i K_k$ *have strictly smaller* ε_i.

Proposition 4. ([GV], Corollary 3.6) *Let* M *be an irreducible* H_n*-module.*

(1) *Let* $\varepsilon = \varepsilon_i(M)$ *and* $N = \widetilde{e}_i^{\varepsilon} M$*. Then for* $0 \leqslant k \leqslant \varepsilon$

$$\widetilde{e}_i^k M = \operatorname{cosoc} \operatorname{Ind} N \boxtimes q^i K_{\varepsilon - k}.$$

(2) $\widetilde{e}_i M \simeq \operatorname{cosoc} e_i M$.

(3) *If* L *is also irreducible and* $\widetilde{e}_i M \simeq \widetilde{e}_i L \neq 0$*, then* $M \simeq L$.

Corollary 5.2. *Let* M *be an irreducible* H_n*-module and let* $\varepsilon = \varepsilon_i(M)$*. Then* $e_i^{\varepsilon} M$ *is the direct sum of* $\varepsilon!$ *copies of* $\widetilde{e}_i^{\varepsilon} M$.

Proof. From Proposition 4, if $N = \widetilde{e}_i^{\varepsilon} M$, then $M = \operatorname{cosoc} \operatorname{Ind}_{n-\varepsilon,1,\ldots,1}^n N \boxtimes q^i \boxtimes \cdots \boxtimes q^i = \operatorname{cosoc} \operatorname{Ind}_{n-\varepsilon,\varepsilon}^n N \boxtimes q^i K_\varepsilon$ and $\varepsilon_i(N) = 0$. Applying the exact functor e_i^{ε} (and using Lemma 1 below) yields a surjection

$$\bigoplus\nolimits_1^{\varepsilon} N \twoheadrightarrow e_i^{\varepsilon} M. \tag{5}$$

But by Frobenius reciprocity, we have a map $N \boxtimes q^i K_\varepsilon \to \operatorname{Res}_{n-\varepsilon,\varepsilon}^n M$, which must be an injection by Kato's theorem. Since restriction is exact, this shows (5) must be an isomorphism. $\square$

The next proposition follows from Proposition 4 and [G], Proposition 10.4.

Proposition 5. *Let* M *be an irreducible* H_n*-module such that* $\varepsilon_i(M) > 0$*. Then*

(1) $\varepsilon_i(\widetilde{e}_i M) = \varepsilon_i(M) - 1$.

(2) $\varepsilon_{i\pm1}(\widetilde{e}_i M) = \begin{cases} \varepsilon_{i\pm1}(M) & \textit{or} \\ \varepsilon_{i\pm1}(M) + 1. \end{cases}$

5.1. The shuffle lemma

We write S_n for the symmetric group on n letters, $s_i = (i, \quad i+1)$, $1 \leqslant i \leqslant n-1$, for the simple transpositions. Denote length by $\ell(w)$.

If $P = (a_1, \ldots, a_k)$ is an *ordered* tuple, $a_i \in \mathbb{Z}_{>0}$, $\sum a_i = n$, it is convenient to denote $S_P = S_{a_1} \times \cdots \times S_{a_k} \subseteq S_n$. For example, $S_{(n)} = S_n$, $S_{(1,\ldots,1)} = \{1\}$.

For such P, we write $W_{\min}^P$ for the set of minimal length left coset representatives of $S_P \subseteq S_n$.

The following lemmas are special cases of the "Mackey formula" relating the composite of induction from H_P to H_n with the restriction from H_n to $H_{P'}$, for various P and P'. In particular, in Lemma 1, $P = (a, b)$, $P' = (n-1, 1)$ (but then we further restrict to H_{n-1}); in Lemma 2 below, $P = P' = (1, 1, \ldots, 1)$ so that $H_P = S$; and finally in Lemma 3, $P = (m, n)$ and $P' = (1, 1, \ldots, 1)$.

Below we compute the action of e_i on an induced module. We refer the reader to [G] or [GV] for a proof.

Lemma 1. *Let A be an irreducible H_a-module and B be an irreducible H_b-module. Let $n = a + b$. The following is an exact sequence:*

$$0 \to \mathrm{Ind}_{a,b-1}^{n-1} A \boxtimes e_i B \to e_i(\mathrm{Ind}_{a,b}^{n} A \boxtimes B) \to \mathrm{Ind}_{a-1,b}^{n-1} e_i A \boxtimes B \to 0.$$

Sometimes, we will want to determine the cosocle of the middle term in the above exact sequence. Theorems 3.1 and 3.5 below describe that for very special A and B, we can determine exactly when

$$\widetilde{e}_i(\mathrm{cosoc}\,\mathrm{Ind} A \boxtimes B) = \begin{cases} \mathrm{cosoc}(\mathrm{Ind}\widetilde{e}_i A \boxtimes B), \\ \mathrm{cosoc}(\mathrm{Ind} A \boxtimes \widetilde{e}_i B), \end{cases}$$

and that the cosocles above are irreducible. (The first case happens when $\varepsilon_i(A) > \varphi_i(B)$, and the second when $\varepsilon_i(A) \leqslant \varphi_i(B)$. We will define φ_i in Section 10.)

Compare this to equation (17) in Section 10 concerning a tensor product of crystals.

Lemma 2. *Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \{q^i\}^n$. Then*

$$\mathrm{ch}(\mathrm{Ind}_S^{H_n} \gamma_1 \boxtimes \cdots \boxtimes \gamma_n) = \sum_{w \in S_n} w \cdot \gamma.$$

We now describe the character of $\mathrm{Ind}_P^n M$ in terms of $\mathrm{ch}\, M$. We say a string $\gamma = \gamma_1\gamma_2 \cdots \gamma_k$ is a shuffle of t and u if t is a subword of γ and u is its complementary subword. The shuffle of t and u, denoted $t \sqcup\!\sqcup\, u$, is then the formal sum of all shuffles of t and u, with multiplicity.

The permutations of $\{1, 2, \ldots, n+m\}$ that keep $1, \ldots, m$ in order and $m+1, \ldots, m+n$ in order (i.e., their shuffles) are exactly the minimal length left coset representatives $W_{\mathrm{min}}^{(m,n)}$. It follows that if we write tu for the concatenation of t and u, which have length m and n respectively, then

$$t \sqcup\!\sqcup\, u = \sum_{w \in W_{\mathrm{min}}^{(m,n)}} w \cdot (tu).$$

We extend $\sqcup\!\sqcup$ linearly to sums of words.

Lemma 3. *Let $M \in \mathrm{Rep}\, H_m$, $N \in \mathrm{Rep}\, H_n$. Then*

$$\mathrm{ch}\,\mathrm{Ind}_{(m,n)}^{m+n} M \boxtimes N = \sum_{\substack{\gamma \text{is a shuffle} \\ \text{of } t \text{ and } u}} (\dim M[t] \dim N[u])\gamma = \mathrm{ch}\, M \sqcup\!\sqcup\, \mathrm{ch}\, N.$$

5.2. Induced modules

Proposition 6. *Let A be an irreducible H_a-module and let B be an irreducible H_b-module. Let $n = a + b$. Then $\mathrm{Ind}_{a,b}^{n} A \boxtimes B \simeq \widehat{\mathrm{Ind}}_{b,a}^{n} B \boxtimes A$, where $n = a + b$.*

Proof. A full proof can be found in Proposition 3.5 of [V]. We will outline the construction of an explicit isomorphism.

Let $W = W_{\min}^{(a,b)} = \{w\}$ = minimal length left coset representatives for $S_a \times S_b \subseteq S_n$. Let $W' = \{x_0 w^{-1}\}_{w \in W} = \{w'\}$ = minimal length right coset representatives for $S_b \times S_a \subseteq S_n$, where x_0 is the longest element of W. If $\{u \otimes v\}$ is a basis of $A \boxtimes B$, then $\{v \otimes u\}$ is a basis of $B \boxtimes A$, so $\{T_w \otimes (u \otimes v)\}_{w \in W}$ is a basis of $\mathrm{Ind}_{a,b}^n A \boxtimes B$, and $\{\phi_{w',u\otimes v}\}_{w' \in W'}$ is a basis of $\widehat{\mathrm{Ind}}_{b,a}^n B \boxtimes A = \mathrm{Hom}_{H_{(b,a)}}(H_n, B \boxtimes A)$, where $\phi_{w',u\otimes v}(T_{x'}) = \delta_{w',x'} v \otimes u$ for $x' \in W'$ (and extend to a homomorphism on all of $H_{(b,a)}$).

Write $\phi_{u\otimes v}$ for $\phi_{x_0,u\otimes v}$. Define the isomorphism

$$\begin{aligned} \psi : \mathrm{Ind}_{a,b}^n A \boxtimes B &\longrightarrow \widehat{\mathrm{Ind}}_{b,a}^n B \boxtimes A \\ h \otimes (u \otimes v) &\overset{\psi}{\longmapsto} h\phi_{u\otimes v} \qquad \text{for } h \in H_n. \end{aligned}$$

To check that ψ is well defined, it suffices by Frobenius reciprocity to check that the map $u \otimes v \mapsto \phi_{u\otimes v}$ is an $H_a \otimes H_b$-map. Then one must show that ψ is surjective, yielding it is an isomorphism by a dimension count. We leave these details to the reader. □

Proposition 7. *Let* $N = \mathrm{cosoc}\,\mathrm{Ind}_{a,b}^n A \boxtimes B$ *where* $n = a+b$. *Then* $N \simeq \mathrm{soc}\,\mathrm{Ind}_{b,a}^n B \boxtimes A \simeq \mathrm{soc}\,\widehat{\mathrm{Ind}}_{a,b}^n A \boxtimes B$.

Proof. It is well known that there exists an involution $D : \mathrm{Rep}_q \to \mathrm{Rep}_q$ that takes irreducibles to themselves and

$$0 \to A \to B \to C \to 0 \iff 0 \to D(C) \to D(B) \to D(A) \to 0.$$

This is an unpublished result of Bernstein [B] that has since entered the literature. This functor also has the property that

$$D(\mathrm{Ind} M) = \widehat{\mathrm{Ind}} D(M).$$

In other words, there is an even stronger relationship between Ind and $\widehat{\mathrm{Ind}}$: for M irreducible, the factors in the socle series of $\mathrm{Ind} M$ coincide with the factors in the cosocle series of $\widehat{\mathrm{Ind}} M$ *in order*. □

We remark that where we use Proposition 7 in this paper, it is possible to use alternate arguments using characters, but they are far less elegant.

Proposition 8. *Let* $P = (n_1, \dots, n_k)$ *and* $n = \sum_i n_i$. *Let* γ_i *be a one-dimensional* H_{n_i}-*module, and so we can write* γ_i *for its character as well.* (*In Section* 4.2 *we took* $n_i = 1$.) *If* Q *is any quotient of* $\mathrm{Ind}_P^n \gamma_1 \boxtimes \cdots \boxtimes \gamma_k$, *then* $\mathrm{ch}\, Q$ *contains the concatenation* $\gamma_1\gamma_2 \cdots \gamma_k$. *If* L *is any submodule of* $\mathrm{Ind}_P^n \gamma_1 \boxtimes \cdots \boxtimes \gamma_k$, *then* $\mathrm{ch}\, L$ *contains the term* $\gamma_k \cdots \gamma_2\gamma_1$.

Proof. Frobenius reciprocity gives us that $\mathrm{Hom}_{H_n}(\mathrm{Ind}_P^n \gamma_1 \boxtimes \cdots \boxtimes \gamma_k, Q) = \mathrm{Hom}_{H_P}(\gamma_1 \boxtimes \cdots \boxtimes \gamma_k, \mathrm{Res}_P^n Q)$, from which the first statement is immediate once we restrict from H_P to $S = H_{(1,\dots,1)}$.

Similarly, the second follows by Proposition 6 since $\mathrm{Hom}_{H_n}(L, \mathrm{Ind}_P^n \gamma_1 \boxtimes \cdots \boxtimes \gamma_k) = \mathrm{Hom}_{H_n}(L, \widehat{\mathrm{Ind}}_{P'}^n \gamma_k \boxtimes \cdots \boxtimes \gamma_2 \boxtimes \gamma_1) = \mathrm{Hom}_{H_{P'}}(\mathrm{Res}_{P'}^n L, \gamma_k \boxtimes \cdots \boxtimes \gamma_2 \boxtimes \gamma_1)$. □

Proposition 9. *Let M be an irreducible H_P-module. Suppose γ occurs as a term in* $\mathrm{ch}(\mathrm{Ind}_P^n M)$ *with multiplicity m and also that γ occurs with multiplicity m in the character of any quotient of* $\mathrm{Ind}_P^n M$*. Then* $\mathrm{cosoc}\,\mathrm{Ind}_P^n M$ *is irreducible and occurs with multiplicity one as a composition factor of* $\mathrm{Ind}_P^n M$*. If in addition γ occurs in* $\mathrm{ch}(\mathrm{soc}\,\mathrm{Ind}_P^n M)$*, then* $\mathrm{Ind}_P^n M$ *is irreducible.*

Proof. The first statement follows since the map

$$\mathrm{Ind} M \twoheadrightarrow \mathrm{cosoc}\,\mathrm{Ind} M$$

is also a homomorphism of $S = R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$-modules. Restricting to S, the γ-isotypic component (generalized eigenspace) of $\mathrm{Ind} M$ must surject to that of $\mathrm{cosoc}\,\mathrm{Ind} M$ because taking generalized eigenspaces is exact. Hence γ can occur with multiplicity at most m in the cosocle. By hypothesis, each irreducible component of $\mathrm{cosoc}\,\mathrm{Ind} M$ accounts for m copies of γ, and so the cosocle must be irreducible. A similar counting argument shows it must occur with multiplicity one as a composition factor of $\mathrm{Ind} M$.

Further, since the cosocle accounts for all m copies of module γ, if γ also occurs in $\mathrm{ch}(\mathrm{soc}\,\mathrm{Ind} M)$, then $\mathrm{soc}\,\mathrm{Ind} M \supseteq \mathrm{cosoc}\,\mathrm{Ind} M$. Hence, $\mathrm{cosoc}\,\mathrm{Ind} M$ is a submodule and we can consider the quotient $\mathrm{Ind} M/(\mathrm{cosoc}\,\mathrm{Ind} M)$ whose character cannot have any copies of γ and so must be zero. Thus $\mathrm{Ind} M = \mathrm{cosoc}\,\mathrm{Ind} M$ is irreducible (by the first statement). $\square$

Remark 4. The same argument as above also shows that if $\mathrm{cosoc}\,\mathrm{Ind} M$ is irreducible, occurs with multiplicity one as a composition factor of $\mathrm{Ind} M$, and also occurs in the socle, then $\mathrm{Ind} M$ is irreducible.

6. More theorems on multisegments

This section is devoted to recalling results from [BZ, Z] on multisegments, translated into the language of Hecke algebras from that of p-adic $GL(n)$. (See also [Ro].) For convenience to the reader, we give complete proofs. We end by giving the significance of these results in the language of crystal graphs.

Recall if $i \leqslant j$, then $\triangle_{(i,j)} = (q^i q^{i+1} \cdots q^j)$ is the one dimensional H_{j-i+1}-module on which all $T_k - q$ and all $X_k - q^{k+i-1}$ vanish.

Observe that $e_j \triangle_{(i,j)} = \widetilde{e}_j \triangle_{(i,j)} = \triangle_{(i,j-1)}$, (with convention $\triangle_{(i,i-1)} = \mathbf{1}$), $e_i \mathbf{1} = 0$, and if $k \neq j$, then $e_k \triangle_{(i,j)} = \widetilde{e}_k \triangle_{(i,j)} = 0$.

6.1. Linking rule

Lemma 4. ([BZ, Z] The linking rule)

(i) $\mathrm{Ind}\triangle_{(i,j)} \boxtimes \triangle_{(k,l)}$ *is irreducible if* $j + 1 < k$.

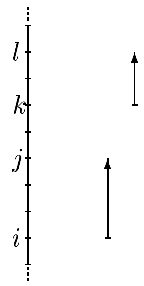
(ii) $\mathrm{Ind}\triangle_{(i,l)} \boxtimes \triangle_{(j,k)}$ *is irreducible if* $i \leqslant j \leqslant k \leqslant l$.

(iii) $N = \mathrm{cosoc}\,\mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)}$ *is irreducible if* $i < j, k < l, j \leqslant k + 1$. *Moreover, the following*

$$0 \to \mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(i,l)} \to \mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)} \to N \to 0 \qquad (6)$$

is exact. In this case, we say $\triangle_{(i,k)}$ *and* $\triangle_{(j,l)}$ *are linked or are a linked pair. If* $j = k + 1$, *then* $\triangle_{(i,k)}$ *and* $\triangle_{(j,l)}$ *are adjacent.*

Proof. (i) This is the case that the intervals $[ij]$ and $[kl]$ are far apart, so that their union is not again an interval.

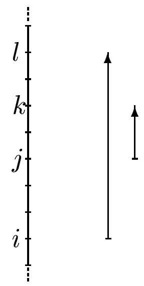


Let $M = \mathrm{Ind}^n_{j-i+1,l-k+1} \triangle_{(i,j)} \boxtimes \triangle_{(k,l)}$ where $n = j+l-i-k+2$. By the shuffle lemma,

$$\mathrm{ch}\, M = \sum_{w \in W^{(j-i+1,l-k+1)}_{\min}} w \cdot (q^i \cdots q^j q^k \cdots q^l). \tag{7}$$

Frobenius reciprocity implies that the term $q^i \cdots q^j q^k \cdots q^l$ must occur in any quotient of M, but by Remark 3 all of the $\binom{n}{j-i+1}$ terms in (7) must occur as well, yielding M that is irreducible.

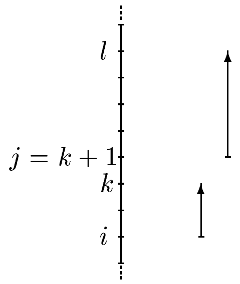
(ii) This is the case that one interval $[jk] \subseteq [il]$ is contained in the other.



Let $n = l-i+k-j+2$ and $L = \mathrm{Ind}^n_{l-i+1,k-j+1} \triangle_{(i,l)} \boxtimes \triangle_{(j,k)}$ where $i \leqslant j \leqslant k \leqslant l$. By Frobenius reciprocity the term $q^i q^{i+1} \cdots q^l q^j q^{j+1} \cdots q^k$ must occur in $\mathrm{ch}(\mathrm{cosoc}\, L)$. The term $q^i \cdots q^j q^j q^{j+1} q^{j+1} \cdots q^k q^k \cdots q^l$ must also occur in $\mathrm{ch}(\mathrm{cosoc}\, L)$ by Remark 3 along with the shuffle lemma, and Kato's theorem (Theorem 5.1) implies it occurs with multiplicity at least $2^{(k-j+1)}$. On the other hand, the shuffle lemma shows this is its multiplicity in $\mathrm{ch}\, L$. Similar reasoning shows this term occurs in $\mathrm{ch}(\mathrm{soc}\, L)$. By Proposition 9, $L = \mathrm{Ind} \triangle_{(i,l)} \boxtimes \triangle_{(j,k)}$ is irreducible.

(iii) This is the case that the union of the intervals $[jl] \cup [ik]$ is a longer interval (in $\mathbb{Z}$). Let $n = l-j+k-i+2$ and $N = \mathrm{cosoc}\, \mathrm{Ind} \triangle_{(j,l)} \boxtimes \triangle_{(i,k)}$ where $i < j, k < l, j \leqslant k+1$.

First we will consider the case that $j = k+1$, that is $\triangle_{(j,l)}$ and $\triangle_{(i,k)}$ are an adjacent pair.



We want to show that

$$0 \to \triangle_{(i,l)} \to \mathrm{Ind}\triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)} \to N \to 0 \tag{8}$$

is an exact sequence and that N is irreducible. By Frobenius reciprocity, because $\mathrm{Res}^n_{l-k,k-i+1} \triangle_{(i,l)} = \triangle_{(i,k)} \boxtimes \triangle_{(k+1,l)}$ there is a nonzero map

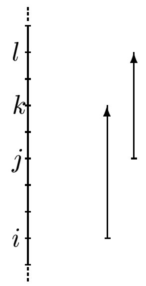
$$\triangle_{(i,l)} \to \widehat{\mathrm{Ind}}^n_{k-i+1,l-k} \triangle_{(i,k)} \boxtimes \triangle_{(k+1,l)} \simeq \mathrm{Ind}^n_{l-k,k-i+1} \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)}.$$

Further as $q^i \cdots q^l$ occurs only once in $\mathrm{ch}\,\mathrm{Ind}^n_{k-i+1,l-k} \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)}$, the socle of the induced module (in (8)) must be $\triangle_{(i,l)}$. (Note that Frobenius reciprocity (Proposition 8) shows this one-dimensional submodule cannot be in the cosocle.) The quotient by this submodule has character $\sum_{\substack{w \in W_{\min}^{(k-i+1,l-k+1)} \\ w \neq 1}} w \cdot q^i \cdots q^l$, any one term of which (in particular $q^{k+1} \cdots q^l q^i \cdots q^k$) ensures the occurrence of the other terms, by Remark 3. This implies the quotient is irreducible and hence must equal N. By the same reasoning

$$0 \to N \to \mathrm{Ind}\triangle_{(i,k)} \boxtimes \triangle_{(k+1,l)} \to \triangle_{(i,l)} \to 0 \tag{9}$$

is exact.

Now consider a linked pair that is not adjacent, so that $i < j \leqslant k < l$.



Using (8) and the exactness of induction we have an injection

$$0 \to \mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(i,l)} \xrightarrow{\alpha} \mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)} \tag{10}$$

and from (9) a surjection

$$\mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)} \xrightarrow{\beta} \mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)} \to 0. \tag{11}$$

From the shuffle lemma and the fact that $i < j, k < l$, the term

$$q^j \cdots q^k q^i \cdots q^j \cdots q^k \cdots q^l$$

occurs in each of the characters of $\mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(i,l)}$, $\mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(k+1,l)} \boxtimes \triangle_{(i,k)}$, and $\mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)}$ with multiplicity one. Because α is injective and β is surjective, this implies that $\beta \circ \alpha$ is nonzero on the $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ eigenvector with eigenvalue $(q^j, \ldots, q^k, q^i, \ldots, q^l)$. By part (ii), $\mathrm{Ind}\,\triangle_{(j,k)} \boxtimes \triangle_{(i,l)} = \mathrm{Ind}\, q^j \cdots q^k \boxtimes (q^i \cdots q^j \cdots q^k \cdots q^l)$ is irreducible, and so $\beta \circ \alpha$ is an injection.

Observe that the term $q^j \cdots q^l q^i \cdots q^k$ occurs with multiplicity one in

$$\mathrm{ch}(\mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)})$$

so that its cosocle N is irreducible by Proposition 9. Furthermore N cannot intersect $\mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(i,l)}$ because this term is not in its character. Thus we have that

$$\mathrm{Ind}\triangle_{(j,k)} \boxtimes \triangle_{(i,l)} \subseteq B$$

where B is the kernel of the map

$$0 \to B \to \mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)} \to N \to 0. \tag{12}$$

We want to show that this inclusion is an equality.

Let A be any composition factor of $\mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)}$. We will show $\varepsilon_k(A) = 1$. First, $\varepsilon_k(A) \leqslant 1$ by the shuffle lemma. Let $w \cdot (q^j \cdots q^l q^i \cdots q^k)$ be a term in $\mathrm{ch}\, A$ for some $w \in W_{\min}^{(l-j+1, k-i+1)}$. If $w(n) = n$, we are done.

If $w(n) \neq n - l + k$, then Remark 3 implies the term $s_{n-1} \cdots s_{w(n)+1} s_{w(n)} w \cdot (q^j \cdots q^l q^i \cdots q^k)$ also occurs in $\mathrm{ch}\, A$ and this permutation does fix n. In other words, the irreducible module

$$\mathrm{Ind} q^k \boxtimes \triangle_{(l-n+w(n)+1,l)} = \mathrm{Ind}\triangle_{(k,k)} \boxtimes \triangle_{(l-n+w(n)+1,l)} \simeq \mathrm{Ind}\triangle_{(l-n+w(n)+1,l)} \boxtimes \triangle_{(k,k)}$$

has $\varepsilon_k = 1$, and this module occurs in the restriction $\mathrm{Res}_{n-w(n)}^{w(n), n-w(n)} \mathrm{Res}_{w(n), n-w(n)}^{n} A$.

In the case $w(n) = n - l + k$, let a be such that $w(n - l + k) = n - l + k + 1 + a$. Then $w \cdot (q^j \cdots q^l q^i \cdots q^k) = (\cdots q^k q^{k-a} \cdots q^{k-1} q^k q^{k+1} \cdots q^l)$. Repeated application of Remark 3 allows us to “slide” the leftmost q^k over so that $\cdots q^{k-1} q^k \cdots q^l q^k$ also occurs in $\mathrm{ch}\, A$, yielding $\varepsilon_k(A) = 1$.

Now we induct on $k - j + 1$, the case $k - j + 1 = 0$ already completed for adjacent pairs above. Assume $k - j + 1 > 0$. Starting with (12), we apply the exact functor e_k yielding

$$0 \to e_k B \to \mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k-1)} \to e_k N \to 0. \tag{13}$$

The middle term is $e_k \mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k)}$ by Lemma 1. Because $k - j + 1 > 0$, the pair $\triangle_{(j,l)}, \triangle_{(i,k-1)}$ is still linked, and by the inductive hypothesis we know its two composition factors. Because $\varepsilon_k(N) = 1$, Proposition 4 implies $e_k N = \widetilde{e}_k N$ is irreducible, and by the inductive hypothesis it must be equal to $\mathrm{cosoc}\, \mathrm{Ind}\triangle_{(j,l)} \boxtimes \triangle_{(i,k-1)}$. From the inductive hypothesis and the exactness of (13), we must have that $e_k B = \mathrm{Ind}\triangle_{(j,k-1)} \boxtimes \triangle_{(i,l)}$, which is irreducible by part (ii) of this lemma. If B were reducible, then because all composition factors of B have $\varepsilon_k \neq 0$, $e_k B$ would also be reducible. Hence we must have $B = \mathrm{Ind}\triangle_{(j,k)} \triangle_{(i,l)}$. $\square$

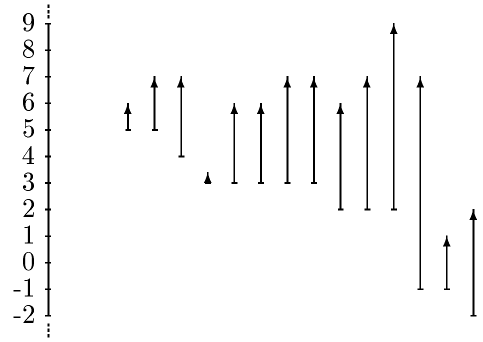
6.2. Classification

Recall that a multiset of segments is a *multisegment*. Theorem 2.2 below is the result from [BZ, Z] that multisegments parameterize the irreducible modules of $\mathrm{Rep}_q\, H_n^{\mathrm{aff}}$.

We also remind the reader of the two orderings introduced in Section 2, and include examples.

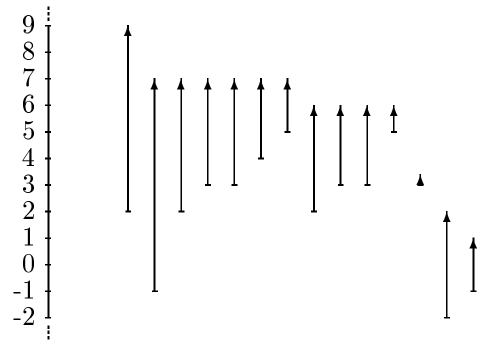
Right order: $\triangle_{(i_1,j_1)} > \triangle_{(i_2,j_2)}$ if $i_1 > i_2$ or if $i_1 = i_2$ and $j_2 > j_1$.

The following picture shows segments $\{\triangle_{(5,6)}, \triangle_{(5,7)}, \triangle_{(4,7)}, \triangle_{(3,3)}, \triangle_{(3,6)}, \triangle_{(3,6)}, \triangle_{(3,7)}, \triangle_{(3,7)}, \triangle_{(2,6)}, \triangle_{(2,7)}, \triangle_{(2,9)}, \triangle_{(-1,7)}, \triangle_{(-1,1)}, \triangle_{(-2,2)}\}$ in the right order, weakly decreasing from left to right.



Left order: $\triangle_{(i_1,j_1)} > \triangle_{(i_2,j_2)}$ if $j_1 > j_2$ or if $j_1 = j_2$ and $i_2 > i_1$.

The following picture shows the *same* segments as above in the left order, weakly decreasing from left to right. In other words in the order $\{\triangle_{(2,9)}, \triangle_{(-1,7)}, \triangle_{(2,7)}, \triangle_{(3,7)}, \triangle_{(3,7)}, \triangle_{(4,7)}, \triangle_{(5,7)}, \triangle_{(2,6)}, \triangle_{(3,6)}, \triangle_{(3,6)}, \triangle_{(5,6)}, \triangle_{(3,3)}, \triangle_{(-2,2)}, \triangle_{(-1,1)}\}$. (Notice that if one rotates a picture of segments in the right order by 180°, one gets a picture of *different* segments in the left order. However, this observation is useful in seeing why theorems for e_i also hold for $\widehat{e}_i$. In fact, if we think of the above as a picture of $\mathrm{ch}\,\mathrm{Ind}\boldsymbol{\Delta}$, then the rotated picture (with arrows pointing down) would be of $\mathrm{ch}(\mathrm{rev}^* M)$.)



If we say $\boldsymbol{\Delta}$ is in one of the orders above, we mean its segments are weakly decreasing from left to right.

Given a multisegment $\boldsymbol{\Delta}$, let $\boldsymbol{\Delta}$ denote the multisegment in the right order, and $\underline{\boldsymbol{\Delta}}$ denote it in the left order. We use the right order when computing $\widetilde{e}_i$ and the left order when computing $\widehat{\widetilde{e}}_i$.

Given a multisegment and a choice of ordering of its segments

$$\mathbf{\Delta} = \{\triangle_{(i_1,j_1)}, \ldots, \triangle_{(i_m,j_m)}\},$$

recall that we let $\mathrm{Ind}\mathbf{\Delta}$ denote $\mathrm{Ind}^n_{n_1,n_2,\ldots,n_m}\triangle_{(i_1,j_1)} \boxtimes \cdots \boxtimes \triangle_{(i_m,j_m)}$, where $n_k = j_k - i_k + 1$ and $n = \sum_k n_k$. Also $n(\mathbf{\Delta}) = n$ and $m(\mathbf{\Delta}) = m$.

Observe that different orderings of a multisegment $\mathbf{\Delta}$ can give non-isomorphic $\mathrm{Ind}\mathbf{\Delta}$. For instance, let $\mathbf{\Delta} = \{\triangle_{(0,0)}, \triangle_{(1,1)}\}$. We know that $\mathrm{Ind}\underline{\mathbf{\Delta}} = \mathrm{Ind}^2_{1,1} 1 \boxtimes q \not\simeq \mathrm{Ind}^2_{1,1} q \boxtimes 1 = \mathrm{Ind}\mathbf{\Delta}$, although they do (always) have the same composition factors.

The corollary below says that the modules $\mathrm{Ind}\mathbf{\Delta}$ are isomorphic to each other for the two orderings above.

Corollary 6.1. *Let* $\mathbf{\Delta}$ *be a multisegment. Then* $\mathrm{Ind}\mathbf{\Delta} \simeq \mathrm{Ind}\underline{\mathbf{\Delta}}$.

Proof. From $\mathbf{\Delta}$ in the right order, consider a segment $\triangle_{(i_0,j_0)}$ with the largest left order, i.e., with largest j, and among those with smallest i. In terms of a picture, that means one (not necessarily unique) that reaches the highest, and is the longest among those. (In the example above, that would be the eleventh segment $\triangle_{(2,9)}$.) Because j_0 is the largest, all segments $\triangle_{(i,j)}$ to the left of $\triangle_{(i_0,j_0)}$ have $j \leqslant j_0$; and because $\mathbf{\Delta}$ is in the right order, we also have $i \geqslant i_0$. In other words $i_0 \leqslant i \leqslant j \leqslant j_0$, so by part (ii) of the linking lemma $\mathrm{Ind}\triangle_{(i,j)} \boxtimes \triangle_{(i_0,j_0)} \simeq \mathrm{Ind}\triangle_{(i_0,j_0)} \boxtimes \triangle_{(i,j)}$. By transitivity of induction, and repeating this argument, the module $\mathrm{Ind}\mathbf{\Delta}$ does not change if we slide $\triangle_{(i_0,j_0)}$ all the way over to the left, where it belongs with respect to the left order. (One can see $\triangle_{(2,9)}$ is indeed at the far left in the example of the left order above.) We can repeat this process with the remaining segments, eventually yielding $\mathrm{Ind}\mathbf{\Delta} \simeq \mathrm{Ind}\underline{\mathbf{\Delta}}$. $\square$

Theorem 2.2 stated in Section 3 shows that multisegments parameterize the irreducible modules in $\mathrm{Rep}_q H_n$, $n \geqslant 0$. In other words, given a multisegment $\mathbf{\Delta}$, the module $M_{\mathbf{\Delta}} = \operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta}$ is irreducible, and as $\mathbf{\Delta}$ ranges over all multisegments, $M_{\mathbf{\Delta}}$ ranges over all irreducibles in Rep_q.

We will include the proof of Theorem 2.2 here, starting with the following lemma.

Lemma 5. *Suppose* $\mathbf{\Delta} = \{\triangle_{(i_1,j)}, \triangle_{(i_2,j)}, \ldots, \triangle_{(i_m,j)}\}$. *Then* $\mathrm{Ind}\mathbf{\Delta}$ *is irreducible.*

Proof. Let $\mathbf{\Delta}^- = \{\triangle_{(i_1,j-1)}, \ldots, \triangle_{(i_m,j-1)}\}$ (entirely omitting segments for which $i_k = j$). By induction, we may assume $\mathrm{Ind}\mathbf{\Delta}^-$ is irreducible. Observe that the base case of this lemma is given by Kato's Theorem, which asserts $\mathrm{Ind} q^j \boxtimes \cdots \boxtimes q^j$ is irreducible.

From the linking lemma, $\mathrm{Ind}\triangle_{(i_1,j-1)} \boxtimes \triangle_{(i_2,j)} \boxtimes \triangle_{(j,j)} \simeq \mathrm{Ind}\triangle_{(i_1,j-1)} \boxtimes \triangle_{(j,j)} \boxtimes \triangle_{(i_2,j)}$ which surjects to $\mathrm{Ind}\triangle_{(i_1,j)} \boxtimes \triangle_{(i_2,j)}$. Iterating this argument, we get a surjection

$$\mathrm{Ind}\mathbf{\Delta}^- \boxtimes q^j \boxtimes q^j \boxtimes \cdots \boxtimes q^j \twoheadrightarrow \mathrm{Ind}\mathbf{\Delta} \twoheadrightarrow \operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta}.$$

Because $\mathrm{Ind}\mathbf{\Delta}^-$ is irreducible by the inductive hypothesis, Proposition 3 shows that $\mathrm{Ind}\mathbf{\Delta}^- \boxtimes q^j \boxtimes \cdots \boxtimes q^j$ has irreducible cosocle M. Therefore we must also have that $\operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta} = M$. Furthermore, M occurs with multiplicity one and is the unique composition factor of $\mathrm{Ind}\mathbf{\Delta}^- \boxtimes q^j \boxtimes \cdots \boxtimes q^j$, and thus of $\mathrm{Ind}\mathbf{\Delta}$, which has $\varepsilon_j(M) = m$ (also by Proposition 3). On the other hand, the linking lemma shows $\mathrm{Ind}\mathbf{\Delta}$ does not depend on the order of its segments, and so by Proposition 7, $M = \operatorname{soc}\mathrm{Ind}\mathbf{\Delta}$ as well. Then by the remark following Proposition 9, $\mathrm{Ind}\mathbf{\Delta} = M$ is irreducible. $\square$

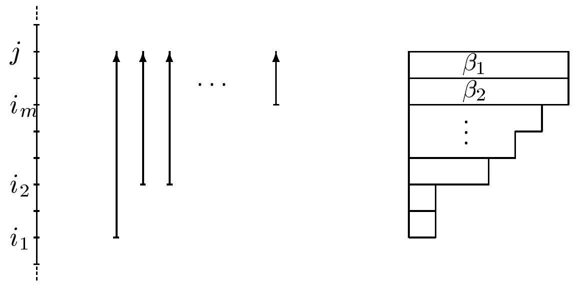
Observe that for $\mathbf{\Delta}$ as in the lemma above, $\varepsilon_j(\mathrm{Ind}\mathbf{\Delta}) = m$, and then by Corollary 5.2 and Lemma 1, $\widetilde{e}_j^m(\mathrm{Ind}\mathbf{\Delta}) = \mathrm{Ind}\mathbf{\Delta}^-$.

Now we complete the proof of Theorem 2.2.

Proof. In light of Corollary 6.1, we may prove the theorem for $M_{\mathbf{\Delta}} = \mathrm{cosoc}\,\mathrm{Ind}\underline{\mathbf{\Delta}}$, which we will do so inducting on n and making use of the functor e_i.

We will show $M_{\mathbf{\Delta}}$ is irreducible. First, we can write $\mathbf{\Delta}$ as a disjoint union of multisegments $\mathbf{\Delta} = \bigcup_j \mathbf{\Delta}^{(j)}$ where $\mathbf{\Delta}^{(j)} = \{\triangle_{(i,k)} \in \mathbf{\Delta} \mid k = j\}$. The lemma above showed $\mathrm{Ind}\mathbf{\Delta}^{(j)}$ is irreducible.

For $\mathbf{\Delta}^{(j)} = \{\triangle_{(i_1,j)}, \ldots, \triangle_{(i_m,j)}\}$ with $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_m$, define $\beta = \beta(\mathbf{\Delta}^{(j)})$ to be the partition conjugate to $(j - i_1 + 1, j - i_2 + 1, \ldots, j - i_m + 1)$.



Let $Q(\mathbf{\Delta}^{(j)})$ be the term $q^{i_1}q^{i_1+1}\cdots q^{i_2}q^{i_2}q^{i_2+1}q^{i_2+1}\cdots q^j q^j \cdots q^j$ and observe that $Q(\mathbf{\Delta}^{(j)})$ occurs with multiplicity $\beta! = \beta_1!\beta_2!\cdots\beta_{j-i_1+1}!$ in $\mathrm{ch}\,\mathrm{Ind}\mathbf{\Delta}^{(j)}$. Also notice that the proof of the previous lemma, the observation made following its proof, and induction show that

$$\mathbf{1} = M_\varnothing = \widetilde{e}_{i_1}^{\beta_{j-i_1+1}} \cdots \widetilde{e}_{j-1}^{\beta_2}\widetilde{e}_j^{\beta_1}(\mathrm{Ind}\mathbf{\Delta}^{(j)}). \tag{14}$$

Now return to $\underline{\mathbf{\Delta}} = \mathbf{\Delta}^{(j_1)} \cup \ldots \cup \mathbf{\Delta}^{(j_t)}$ with $j_1 > j_2 > \ldots > j_t$. Because j is strictly bigger than any exponent occurring in the segments of $\mathbf{\Delta}^{(j_a)}$ if $j > j_a$ from the shuffle lemma, we see that the concatenation $Q(\mathbf{\Delta}^{(j_1)})\cdots Q(\mathbf{\Delta}^{(j_t)})$ occurs as a term in $\mathrm{ch}\,\mathrm{Ind}\underline{\mathbf{\Delta}}$ with multiplicity $\beta(\mathbf{\Delta}^{(j_1)})!\cdots\beta(\mathbf{\Delta}^{(j_t)})!$. This is its multiplicity in $\underline{\mathbf{\Delta}}$ as well. The irreducibility of $\mathrm{Ind}\mathbf{\Delta}^{(j_a)}$ and Frobenius reciprocity show this term occurs in any quotient of $\mathrm{Ind}\underline{\mathbf{\Delta}}$ with that same multiplicity. Therefore by Proposition 9, $M_{\mathbf{\Delta}} = \mathrm{cosoc}\,\mathrm{Ind}\underline{\mathbf{\Delta}} = \mathrm{cosoc}\,\mathrm{Ind}\mathbf{\Delta}$ must be irreducible (and furthermore occurs with multiplicity one in $\mathrm{Ind}\underline{\mathbf{\Delta}}$).

For parts 2 and 3 of the theorem, we give only a brief sketch of the argument from [BZ, Z]. We will see later that it also follows from the computation for $\widetilde{e}_i M_{\mathbf{\Delta}}$.

Next, we will show irreducibles corresponding to distinct multisegments are distinct. Given an irreducible module $M_{\mathbf{\Delta}}$, let j be the *smallest* integer for which $\varepsilon_j(M_{\mathbf{\Delta}}) > 0$. Write $\varepsilon = \varepsilon_j(M_{\mathbf{\Delta}})$. Let $N = \widetilde{e}_j^{\varepsilon} M_{\mathbf{\Delta}}$. From Lemma 5 and the fact that we may assume $\mathbf{\Delta}$ is in the left order, we must have $\varepsilon = m(\mathbf{\Delta}^{(j)}) = m$. Using Lemma 1 and Corollary 5.2, we must have that $N = M_\Gamma$ where

$$\Gamma = (\mathbf{\Delta}^{(j)})^- \cup \textstyle\bigcup_{k \neq j} \mathbf{\Delta}^{(k)}.$$

Notice all the k we union over are bigger than j by our choice of j. Suppose $M_{\mathbf{\Delta}} \simeq M_{\mathbf{\Delta}'}$. Then also $N = \widetilde{e}_j^{\varepsilon} M_{\mathbf{\Delta}'} = M_{\Gamma'}$ for Γ' obtained from $\mathbf{\Delta}'$ in the same way as above. By induction $\Gamma = \Gamma'$. Lemma 5 gives us a well defined and reversible way to go from $\mathbf{\Delta}$ to Γ. By the properties of the left order, if Γ contains k segments of the form $\triangle_{(i,j-1)}$, each one must have come from a $\triangle_{(i,j)}$ in $\mathbf{\Delta}$, and further $\mathbf{\Delta}$ must contain $\varepsilon - k$ segments $\triangle_{(j,j)}$. This holds as well for $\mathbf{\Delta}'$, so that $M_{\mathbf{\Delta}} \simeq M_{\mathbf{\Delta}'} \implies \mathbf{\Delta} = \mathbf{\Delta}'$.

Finally, for part 3, we will show every irreducible module in Rep_q is some $M_{\mathbf{\Delta}}$. Let M be any irreducible module in Rep_q, and let $N = \widetilde{e}_j^{\varepsilon_j(M)} M$, where again j is smallest possible. By induction there is some multisegment Γ such that $N = M_\Gamma$.

We construct a new multisegment $\mathbf{\Delta}$ from Γ by replacing each $\triangle_{(i,j-1)} \in \Gamma$ by $\triangle_{(i,j)}$, and also adding in $\varepsilon_j(M) - \varepsilon_{j-1}(N)$ many segments $\triangle_{(j,j)}$. This number is nonnegative because $\varepsilon_{j-1}(M) = 0$ by choice of j and Proposition 5 shows that $\varepsilon_{j-1}(M) + k \geqslant \varepsilon_{j-1}(\widetilde{e}_j^k(M))$. Then $\mathrm{Ind}\, \underline{\Gamma} \boxtimes q^j \boxtimes \cdots \boxtimes q^j$ has cosocle M, but by an argument similar to that in the proof of Lemma 5, it also surjects to $\mathrm{Ind}\underline{\mathbf{\Delta}}$, which has cosocle $M_{\mathbf{\Delta}}$. Thus $M = M_{\mathbf{\Delta}}$. □

Corollary 6.2. *$M_{\mathbf{\Delta}}$ occurs with multiplicity one as a composition factor of* $\mathrm{Ind}\mathbf{\Delta}$.

Remark 5. Because $M_{\mathbf{\Delta}} = \mathrm{cosoc}\,\mathrm{Ind}\underline{\mathbf{\Delta}}$, if $\triangle_{(i,j)} \in \mathbf{\Delta}$ is smallest in the left order, then j is the smallest integer such that $\varepsilon_j(M_{\mathbf{\Delta}}) \neq 0$ and furthermore

$$\varepsilon_j(M_{\mathbf{\Delta}}) = \mid \{\triangle_{(i,j)} \in \mathbf{\Delta} \mid i \in \mathbb{Z}\} \mid = m(\mathbf{\Delta}^{(j)}).$$

For other j the cardinality of this set is merely an upper bound on $\varepsilon_j(M_{\mathbf{\Delta}})$.

In the proof of parts 2 and 3 of the above theorem, we saw that Lemma 5 allowed us to compute the multisegment Γ for which $\widetilde{e}_j^{\varepsilon_j(M_{\mathbf{\Delta}})} M_{\mathbf{\Delta}} = M_\Gamma$, where j was the *smallest* integer for which $\varepsilon_j(M_{\mathbf{\Delta}}) > 0$. However, at this stage we have not proved that Rule 1 describes $\widetilde{e}_i M_{\mathbf{\Delta}}$ in terms of multisegments for $i \neq j$, nor even $\widetilde{e}_j^k M_{\mathbf{\Delta}}$ when $k \neq \varepsilon_j(M_{\mathbf{\Delta}})$. This will be proved below. However having done so, it will be clear Theorem 3.1 also gives a proof of parts 2 and 3 of Theorem 2.2.

Remark 6. One could now appeal to Grojnowski's Theorem 14.3: that the graph whose nodes are indexed by multisegments $\mathbf{\Delta}$ and whose edges are given by $\Gamma \xrightarrow{i} \mathbf{\Delta}$ if

$$\widetilde{e}_i M_{\mathbf{\Delta}} = M_\Gamma$$

is the crystal graph $B(\infty)$.

Given a multisegment $\mathbf{\Delta}$, we can iterate the rule that we always take $\widetilde{e}_j^{\varepsilon_j}$ for the smallest j such that $\varepsilon_j \neq 0$. Combinatorially, we are just iterating the replacement of $\mathbf{\Delta}^{(j)}$ with $(\mathbf{\Delta}^{(j)})^-$ (if we wrote $\mathbf{\Delta} = \bigcup_k \mathbf{\Delta}^{(k)}$). This constructs a distinguished path on the crystal graph from $\mathbf{\Delta}$ back to the empty multisegment $\varnothing$. Now that we have determined where on the crystal graph the node $\mathbf{\Delta}$ sits, the determination of a single edge

$$? \xrightarrow{i} \mathbf{\Delta}$$

is purely combinatorial. And conversely, we can determine the distinguished path leading from ? back to $\varnothing$, and thus reexpress $\widetilde{\mathbf{e}}_{\mathbf{i}}\mathbf{\Delta}$ as a multisegment. This is exactly given by Rule 1.

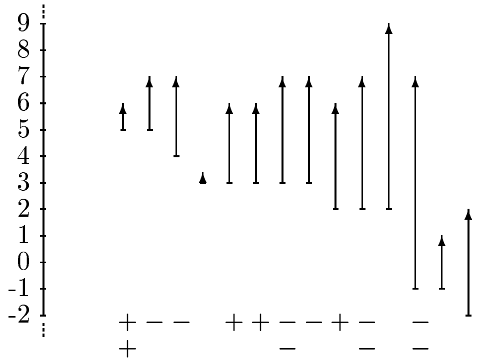
Theorem 3.1 and Rule 1 give the combinatoric that Grojnowski's theorem dictates, but the proof here is module-theoretic. Admittedly, a proof appealing to the known crystal structure described in Theorem 14.3 of [G] is much slicker.

7. Example of computing $\widetilde{\mathbf{e}}_{\mathbf{i}}$

Here we give an example of computing $\widetilde{\mathbf{e}}_{\mathbf{i}}\boldsymbol{\Delta}$.

Let $\boldsymbol{\Delta} = \{\triangle_{(5,6)}, \triangle_{(5,7)}, \triangle_{(4,7)}, \triangle_{(3,3)}, \triangle_{(3,6)}, \triangle_{(3,6)}, \triangle_{(3,7)}, \triangle_{(3,7)}, \triangle_{(2,6)}, \triangle_{(2,7)}, \triangle_{(2,9)}, \triangle_{(-1,7)}, \triangle_{(-1,1)}, \triangle_{(-2,2)}\}$ as in the example for the right order in Section 6.2.

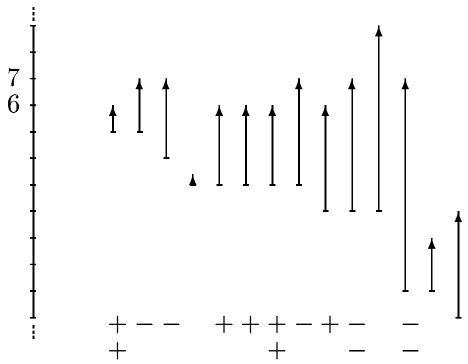
$\boldsymbol{\Delta}$



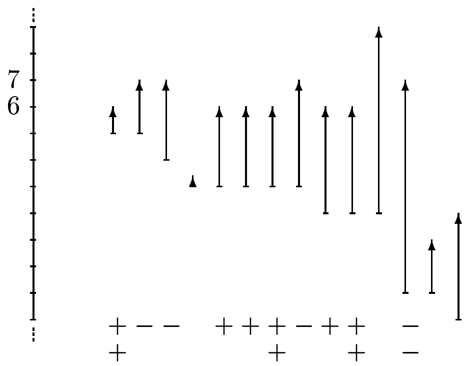
We will compute $\widetilde{\mathbf{e}}_{\mathbf{7}}^{k}\boldsymbol{\Delta}$. First, we compute the word $+--\ ++--+--$ and cancel as $+(-(-+)+)-(-+)--$ leaving $+\quad -\ --$ uncanceled.

Thus $\varepsilon_7(M_{\boldsymbol{\Delta}}) = 3$ and $\widetilde{\mathbf{e}}_{\mathbf{7}}^{4}\boldsymbol{\Delta} = \mathbf{0}$.

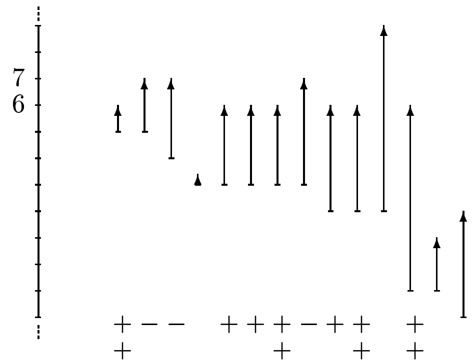
$\widetilde{\mathbf{e}}_{\mathbf{7}}\boldsymbol{\Delta}$



$\widetilde{\mathbf{e}}_{\mathbf{7}}^{2}\boldsymbol{\Delta}$



$\tilde{\mathbf{e}}_7^3\boldsymbol{\Delta}$



As an exercise, the reader can consider $\underline{\boldsymbol{\Delta}}$ and compute $\widehat{\varepsilon}_{-1}(M_{\boldsymbol{\Delta}}) = 2$, $\widehat{\varepsilon}_2(M_{\boldsymbol{\Delta}}) = 2$, $\widehat{\varepsilon}_3(M_{\boldsymbol{\Delta}}) = 4$, $\widehat{\varepsilon}_4(M_{\boldsymbol{\Delta}}) = 1$, $\widehat{\varepsilon}_5(M_{\boldsymbol{\Delta}}) = 2$.

8. Proof of Theorem 3.1

Remark 7. In this section and the next, we will repeatedly use the fact that if a module $\operatorname{cosoc}\operatorname{Ind}A \boxtimes B$ is irreducible, then it coincides with $\operatorname{cosoc}(\operatorname{Ind}(\operatorname{cosoc} A) \boxtimes B)$ and $\operatorname{cosoc}(\operatorname{Ind}A \boxtimes (\operatorname{cosoc} B))$.

Lemma 6. (Sliding lemma, [MW], II.8)

(1) *Suppose $a < b \leqslant c$ and let $\boldsymbol{\Delta} = \{\triangle_{(b,j)}, \triangle_{(a,j-1)}\}$. Then*

$$\operatorname{Ind}M_{\boldsymbol{\Delta}} \boxtimes \triangle_{(c,j)} \simeq \operatorname{Ind}\triangle_{(c,j)} \boxtimes M_{\boldsymbol{\Delta}}$$

is irreducible.

(2) *Let $\Gamma = \{\triangle_{(a,z_0)}, \triangle_{(a-1,z_1)}, \ldots, \triangle_{(a-k,z_k)}\}$, where $z_0 > z_1 > \ldots > z_k \geqslant z$. Then*

$$\operatorname{Ind}M_\Gamma \boxtimes \triangle_{(a-k,z)} \simeq \operatorname{Ind}\triangle_{(a-k,z)} \boxtimes M_\Gamma$$

is irreducible.

Proof. Write $M = M_{\boldsymbol{\Delta}}$. Let $N = \operatorname{cosoc}\operatorname{Ind}\triangle_{(c,j)} \boxtimes M$. Then the segments $\triangle_{(c,j)}, \triangle_{(b,j)}$, $\triangle_{(a,j-1)}$ are in the right order, so that by Remark 7 above, $N = M_{\{\triangle_{(c,j)}, \triangle_{(b,j)}, \triangle_{(a,j-1)}\}}$.

First, we claim $\varepsilon_j M = 0$ and $\varepsilon_j N = 1$. We have an exact sequence

$$0 \to \operatorname{Ind}\triangle_{(b,j-1)} \boxtimes \triangle_{(a,j)} \to \operatorname{Ind}\triangle_{(b,j)} \boxtimes \triangle_{(a,j-1)} \to M \to 0$$

by the linking lemma. Applying the exact functor e_j and using Lemma 1 yields

$$0 \to \operatorname{Ind}\triangle_{(b,j-1)} \boxtimes \triangle_{(a,j-1)} \to \operatorname{Ind}\triangle_{(b,j-1)} \boxtimes \triangle_{(a,j-1)} \to e_j M \to 0.$$

Because the first two terms are irreducible and isomorphic by the linking lemma, the last term must be zero and thus $\varepsilon_j M = 0$.

By the shuffle lemma, $\varepsilon_j N \leqslant 1$. On the other hand, since

$$\operatorname{Ind}\triangle_{(c,j)} \boxtimes \triangle_{(b,j)} = \operatorname{Ind}\triangle_{(b,j)} \boxtimes \triangle_{(c,j)},$$

we know by Proposition 8 that $\operatorname{ch} N$ has a term $q^b q^{b+1} \cdots q^c q^c q^{c+1} q^{c+1} \cdots q^j q^j q^a \cdots q^{j-1}$ and therefore also $q^b \cdots q^j q^j q^{j-1}$, so finally a term $q^b \cdots q^j q^{j-1} q^j$ by Remark 3. Thus $\varepsilon_j N \geqslant 1$.

Because $\varepsilon_j N = 1$, $e_j N = \widetilde{e}_j N$ is irreducible (and nonzero) and we have by Lemma 1

$$\operatorname{Ind}\triangle_{(c,j-1)} \boxtimes M \twoheadrightarrow e_j N = \widetilde{e}_j N,$$

which also shows that $\widetilde{e}_j N = M_{\{\triangle_{(c,j-1)},\triangle_{(b,j)}\triangle_{(a,j-1)}\}}$ since these segments are in the right order.

Let $L = \operatorname{cosoc} \operatorname{Ind} M \boxtimes \triangle_{(c,j)}$. Then as above we get

$$\operatorname{Ind} M \boxtimes \triangle_{(c,j-1)} \twoheadrightarrow e_j L = \widetilde{e}_j L.$$

Now $\triangle_{(b,j)}, \triangle_{(a,j-1)}, \triangle_{(c,j-1)}$ are in the left order, so by Theorem 2.2 and Corollary 6.1, $\widetilde{e}_j L \simeq M_{\{\triangle_{(b,j)},\triangle_{(a,j-1)},\triangle_{(c,j-1)}\}} \simeq M_{\{\triangle_{(c,j-1)},\triangle_{(b,j)},\triangle_{(a,j-1)}\}} \simeq \widetilde{e}_j N$. (In particular, this shows L was irreducible, since any summands of it must have $\varepsilon_j = 1$ as well.) Therefore, from Proposition 4 we know $L \simeq N$.

On the other hand, $\operatorname{cosoc} \operatorname{Ind}\triangle_{(c,j)} \boxtimes M = N \simeq L = \operatorname{soc} \operatorname{Ind}\triangle_{(c,j)} \boxtimes M$ by Proposition 7. However by Corollary 6.2, N occurs with multiplicity one in $\operatorname{Ind}\triangle_{(c,j)} \boxtimes \triangle_{(b,j)} \boxtimes \triangle_{(a,j-1)}$ and therefore in $\operatorname{Ind}\triangle_{(c,j)} \boxtimes M$. From the remark following Proposition 9, $\operatorname{Ind}\triangle_{(c,j)} \boxtimes M$ is irreducible, and thus also isomorphic to $\operatorname{Ind} M \boxtimes \triangle_{(c,j)}$.

This proves part 1. The proof of part 2 is quite similar, but we use $\widehat{e}_i$ instead of e_i. Notationally, it is slightly messier, but it is more useful in this form for use in future theorems.

Now rewrite notation, letting $M = M_\Gamma$ and let $N = \operatorname{cosoc} \operatorname{Ind} M_\Gamma \boxtimes \triangle_{(a-k,z)}$. Write $\Gamma' = \{\triangle_{(a,z_0)}, \ldots, \triangle_{(a-k,z_k)}, \triangle_{(a-k,z)}\}$. Then Γ' is in the left order, so that $N = M_{\Gamma'}$ is irreducible.

The same argument as in part 1 shows $\widehat{\varepsilon}_{a-i}(M_\Gamma) = 0$ if $i > 0$, and so $\widehat{\varepsilon}_{a-k}(N) \leqslant 1$. An argument similar to that in part 1 also shows that

$$\operatorname{Ind} M_{\{\triangle_{(a-k+1,z_{k-1})},\triangle_{(a-k,z_k)}\}} \boxtimes \triangle_{(a-k,z)} \simeq \operatorname{Ind}\triangle_{(a-k,z)} \boxtimes M_{\{\triangle_{(a-k+1,z_{k-1})},\triangle_{(a-k,z_k)}\}}.$$

Together with Remark 3 and Proposition 8, this shows $\widehat{\varepsilon}_{a-k}(N) = 1$.

Let $L = \operatorname{cosoc} \operatorname{Ind}\triangle_{(a-k,z)} \boxtimes M_\Gamma$. Observe that $\operatorname{Ind} M_\Gamma \boxtimes \triangle_{(a,z)} = \operatorname{Ind}\triangle_{(a,z)} \boxtimes M_\Gamma$ by Corollary 6.1 if $a \leqslant z$. Let $b = \min\{a-1, z\}$, so that $\widehat{e}_b \widehat{e}_{b+1} \cdots \widehat{e}_{a-k} \triangle_{(a-k,z)} = \triangle_{(a,z)}$, where we interpret $\triangle_{(a,z)} = \triangle_\varnothing = \mathbf{1}$ if $a > z$. (If $a > z$, then $b = z$ and interpreting $\triangle_{(a,z)} = \mathbf{1}$, the observation still holds.) By repeated application of the exact functor $\widehat{e}_i$ and the analogue to Lemma 1, the cosocle of the above module is equal to each of the following: $\widehat{\widetilde{e}}_b \cdots \widehat{\widetilde{e}}_{a-k+1} \widehat{\widetilde{e}}_{a-k} L = \widehat{e}_b \cdots \widehat{e}_{a-k} L = \operatorname{cosoc} \operatorname{Ind}\triangle_{(a,z)} \boxtimes M_\Gamma = \widehat{e}_b \cdots \widehat{e}_{a-k} N = \widehat{\widetilde{e}}_b \cdots \widehat{\widetilde{e}}_{a-k} N$. That the right hand side is nonzero is similar to the computation that $\widehat{\varepsilon}_{a-k}(N) \geqslant 1$. Proposition 4 implies that $L \simeq N$. A repeat of the argument that concludes part 1 shows that $\operatorname{Ind} M_\Gamma \boxtimes \triangle_{(a-k,z)} \simeq \operatorname{Ind}\triangle_{(a-k,z)} \boxtimes M_\Gamma$ is irreducible. $\square$

The previous lemma is the final step toward computing $\widetilde{e}_j M_{\mathbf{\Delta}}$ where $\mathbf{\Delta} = \mathbf{\Delta}^{(j)} \cup \mathbf{\Delta}^{(j-1)}$, after which we can compute $\widetilde{e}_j M_{\mathbf{\Delta}}$ for any $M_{\mathbf{\Delta}}$. Then we shall show it coincides with Rule 1.

Lemma 7. *For* $\mathbf{\Delta} = \mathbf{\Delta}^{(j)} \cup \mathbf{\Delta}^{(j-1)}$,

$$\widetilde{e}_j M_{\mathbf{\Delta}} = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}}$$

and $\widetilde{e}_j M_{\mathbf{\Delta}} = 0$ *if* $\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta} = \mathbf{0}$. *In particular,* $\varepsilon_j(M_{\mathbf{\Delta}})$ *is the number of* $-$ *signs left after all cancelling is done as in Rule* 1.

Proof. First we prove the second statement. We have

$$M_{\mathbf{\Delta}} = \operatorname{cosoc} \operatorname{Ind}\mathbf{\Delta} = \operatorname{cosoc} \operatorname{Ind}\triangle_{(a_1,\pm)} \boxtimes \cdots \boxtimes \triangle_{(a_m,\pm)},$$

writing $-$ for j and $+$ for $j-1$. The a_{i_k} weakly decrease since we take $\mathbf{\Delta}$ in right order. Recall from the proof of the sliding Lemma 6 that $\varepsilon_j(M_{\{\triangle_{(a_{i-1},j)},\triangle_{(a_i,j-1)}\}}) = 0$ if $a_{i-1} > a_i$, in other words, if the pair is linked. So, using Remark 7, whenever you see an adjacent $\triangle_{(a_{i-1},-)} \boxtimes \triangle_{(a_i,+)}$, it does not contribute to $\varepsilon_j(M_{\mathbf{\Delta}})$. In other words, we can replace each $-+$ by $M_i = M_{\{\triangle_{(a_{i-1},j)},\triangle_{(a_i,j-1)}\}}$, yielding $M_{\mathbf{\Delta}} = \operatorname{cosoc} \operatorname{Ind}\triangle_{(a_1,\pm)} \boxtimes \cdots \boxtimes M_i \boxtimes \cdots \triangle_{(a_m,\pm)}$. Repeat. Then by the sliding lemma, a $\triangle_{(a_{i-k},-)}$ to the left of M_i can *slide* past it, leaving the cosocle of the induced module unchanged.

Then we may repeat the first step, which in effect cancels any adjacent $-+$ pairs, and then by the sliding lemma continues to cancel newly created $-+$ pairs from the remaining symbols, and so on. In the end we are left with something that looks like

$$M_{\mathbf{\Delta}} = \operatorname{cosoc} \operatorname{Ind} \underbrace{\triangle_{(a_1,+)} \cdots \boxtimes M_i \cdots \boxtimes \triangle_{(a_k,+)}}_{A} \boxtimes \underbrace{\triangle_{(a_{k+1},-)} \boxtimes \cdots \boxtimes \triangle_{(a_m,-)}}_{B} \tag{15}$$

where A is an induced module consisting solely of $\triangle_{(a_i,+)}$'s and M_i's (which have the form $M_{\{\triangle_{(a_{i-k},-)},\triangle_{(a_i,+)}\}}$), and B is induced from $\triangle_{(a_i,-)}$'s. Thus $\varepsilon_j(A) = 0$ and $\varepsilon_j(B) = \varepsilon_j(M_{\mathbf{\Delta}})$ is the number of uncanceled $-$ signs remaining from the procedure. If there are no uncanceled $-$ signs left, then it is clear $\varepsilon_j(M_{\mathbf{\Delta}}) = 0$, so that $\widetilde{e}_j M_{\mathbf{\Delta}} = 0 = M_{\mathbf{0}} = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}}$. This proves the second statement. Notice that it was very important that $\mathbf{\Delta}$ was in the right order (and not the left order) to start with.

However, we have also shown that if we alter the right order by sliding segments from left to right past linked pairs, uncanceled $\pm$ symbols will be unchanged.

Now we can determine $\widetilde{e}_j M_{\mathbf{\Delta}}$. Let $\varepsilon = \varepsilon_j(M_{\mathbf{\Delta}})$. We know $M_{\mathbf{\Delta}} = \operatorname{cosoc} \operatorname{Ind}A \boxtimes \triangle_{(a_{i_1},j)} \boxtimes \cdots \boxtimes \triangle_{(a_{i_\varepsilon},j)}$, and $\varepsilon_j(A) = 0$. Lemma 1 tells us that $e_j(\operatorname{Ind}A \boxtimes \triangle_{(a_{i_1},j)} \boxtimes \cdots \boxtimes \triangle_{(a_{i_\varepsilon},j)})$ is filtered by modules whose successive quotients have the form

$$\operatorname{Ind}A \boxtimes \triangle_{(a_{i_1},j)} \boxtimes \cdots \boxtimes \triangle_{(a_{i_s},j-1)} \boxtimes \cdots \boxtimes \triangle_{(a_{i_\varepsilon},j)}, \tag{16}$$

for $s = 1, \ldots, \varepsilon$. Thus $\widetilde{e}_j M_{\mathbf{\Delta}}$ is in the cosocle of one of these. We will show below that each of these induced modules of the form (16) has irreducible cosocle and so the above procedure lets us compute ε_j of those cosocles.

We will show the module in (16) is a quotient of $\operatorname{Ind}\mathbf{\Delta}'$, where $\mathbf{\Delta}' = \mathbf{\Delta} \cup \{\triangle_{(a_{i_s},j-1)}\} \setminus \{\triangle_{(a_{i_s},j)}\}$ (in right order), and so its cosocle is $M_{\mathbf{\Delta}'}$, which is irreducible. Then we may apply the first part of the proof to compute $\varepsilon_j(M_{\mathbf{\Delta}'})$.

Starting with $\operatorname{Ind}\mathbf{\Delta}'$, which we know has irreducible cosocle, we will apply the same moves we performed on $\mathbf{\Delta}$, i.e., ignore $-+$ pairs by forming M_i to end with a module as in (16) as a quotient of $\mathrm{Ind}\mathbf{\Delta}'$. We justify being able to perform these moves as follows. First, by the linking lemma, $\mathrm{Ind}\mathbf{\Delta}'$ is the same whether $\mathbf{\Delta}'$ is in the right order, or in the order inherited from $\mathbf{\Delta}$ in the right order with $\triangle_{(a_{i_s},j)}$ replaced by $\triangle_{(a_{i_s},j-1)}$. So let us start with $\mathbf{\Delta}'$ in the latter order. Observe that when $c < b \leqslant a$, two applications of the linking lemma yield $\mathrm{Ind}\triangle_{(b,j)} \boxtimes \triangle_{(c,j-1)} \boxtimes \triangle_{(a,j-1)} \simeq \mathrm{Ind}\triangle_{(a,j-1)} \boxtimes \triangle_{(b,j)} \boxtimes \triangle_{(c,j-1)}$, the first of which is in the left order and so has irreducible cosocle. Therefore by Remark 7,

$$\operatorname{cosoc}\mathrm{Ind}M_{\{\triangle_{(b,j)},\triangle_{(c,j-1)}\}} \boxtimes \triangle_{(a,j-1)} \simeq \operatorname{cosoc}\mathrm{Ind}\triangle_{(a,j-1)} \boxtimes M_{\{\triangle_{(b,j)},\triangle_{(c,j-1)}\}}.$$

If we compare this to the sliding lemma (Lemma 6), it says any time we slid $\triangle_{(a,j)}$ past an M_i in the above process for $\mathbf{\Delta}$ (cancelling a $-+$), we would also have been allowed to slide the segment past if it were replaced by $\triangle_{(a,j-1)}$. So, starting from $\mathbf{\Delta}'$ we form the same M_i and do the same slidings to end with the module in equation (16) as a quotient of $\mathrm{Ind}\mathbf{\Delta}'$. Hence

$$M_{\mathbf{\Delta}'} = \operatorname{cosoc}\mathrm{Ind}A \boxtimes \triangle_{(a_{i_1},j)} \boxtimes \cdots \boxtimes \triangle_{(a_{i_s},j-1)} \boxtimes \cdots \boxtimes \triangle_{(a_{i_\varepsilon},j)}.$$

In particular, the cosocle is irreducible.

Now we may apply what we have already proved to compute $\varepsilon_j(M_{\mathbf{\Delta}'}) \leqslant \varepsilon - 2$ if $s \neq 1$.

However, we know from Proposition 5 that $\varepsilon_j(\widetilde{e}_j M_{\mathbf{\Delta}}) = \varepsilon - 1$, so we must have $\widetilde{e}_j M_{\mathbf{\Delta}}$ occurring only as the cosocle of the induced module as in (16) with $s = 1$. In other words, it is the *leftmost* remaining (uncanceled) $-$ sign that signifies which segment to alter in computing $\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}$. This process matches Rule 1, and so proves the lemma. (The only difference in (15) and Rule 1 is that we slid $-+$ (i.e., linked) pairs from the right to the left of some $-$ signs. However it is the uncanceled symbols and the segments they correspond to that are used in both computations, identically.) $\square$

We will now use Lemma 7 to complete the proof of Theorem 3.1, that $\widetilde{e}_j M_{\mathbf{\Delta}} = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}}$.

Proof. From Corollary 6.1, we know that

$$M_{\mathbf{\Delta}} = \operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta} = \operatorname{cosoc}\mathrm{Ind}\underline{\mathbf{\Delta}} = \operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta}^{(k_1)} \boxtimes \cdots \boxtimes \mathbf{\Delta}^{(j)} \boxtimes \mathbf{\Delta}^{(j-1)} \boxtimes \cdots \boxtimes \mathbf{\Delta}^{(k_s)}.$$

Let $j = k_t, j-1 = k_{t+1}$. Let

$$A = \operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta}^{(k_1)} \boxtimes \cdots \boxtimes \mathbf{\Delta}^{(k_{t-1})} \quad \text{and} \quad B = \operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta}^{(k_{t+2})} \boxtimes \cdots \boxtimes \mathbf{\Delta}^{(k_s)},$$

so that by Remark 7

$$M_{\mathbf{\Delta}} = \operatorname{cosoc}\mathrm{Ind}A \boxtimes \mathbf{\Delta}^{(j)} \boxtimes \mathbf{\Delta}^{(j-1)} \boxtimes B.$$

Clearly $\varepsilon_j(A) = 0$ and $\varepsilon_j(B) = 0$ by the shuffle lemma. Further, all terms in $\operatorname{ch} B$ have support consisting of q^k where $k < j-1$. Then Remark 3 implies $\mathrm{Ind}q^j \boxtimes B \simeq \mathrm{Ind}B \boxtimes q^j$ (which furthermore is irreducible). In other words, $\varepsilon_j(M_{\mathbf{\Delta}}) = \varepsilon_j(\operatorname{cosoc}\mathrm{Ind}\mathbf{\Delta}^{(j)} \boxtimes \mathbf{\Delta}^{(j-1)})$ and we have reduced Theorem 3.1 to Lemma 7, where the same argument holds. To compute $\widetilde{e}_j$ we now consider $\mathbf{\Delta}^{(j)} \cup \mathbf{\Delta}^{(j-1)}$ in the right order, and indeed this coincides with computing the $-+$ rule on all of $\mathbf{\Delta}$ in the right order because the other segments in $\mathbf{\Delta}^{(k)}$ only contribute blanks. Using Remark 7 and the fact from the proof of Lemma 7 that $\widetilde{e}_j M_{\mathbf{\Delta}^{(j)} \cup \mathbf{\Delta}^{(j-1)}}$ is the only composition factor of $e_j M_{\mathbf{\Delta}^{(j)} \cup \mathbf{\Delta}^{(j-1)}}$ with $\varepsilon_j = \varepsilon_j(M_{\mathbf{\Delta}}) - 1$, we get

$$\widetilde{e}_j M_{\mathbf{\Delta}} = \operatorname{cosoc}\mathrm{Ind}A \boxtimes (\widetilde{e}_j M_{\mathbf{\Delta}^{(j)} \cup \mathbf{\Delta}^{(j-1)}}) \boxtimes B = \operatorname{cosoc}\mathrm{Ind}(\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}) = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}}. \quad \square$$

The proof of Theorem 3.2, that $\widehat{e}_i M_{\mathbf{\Delta}} = M_{\widehat{\mathbf{e}}_{\mathbf{i}}\mathbf{\Delta}}$, is analogous to the proof of Theorem 3.1, and so will not be included.

As a result, we also get Corollary 3.3, which uses Rule 2 to determine which $M_{\mathbf{\Delta}}$ are in $\operatorname{Rep} H_n^{\lambda}$. In the next section we will study the case $\lambda = \Lambda_i$.

9. Multipartitions

In this section, we will combine Remark 7 with Lemma 6 to build up $M_{\mathbf{\Delta}}$ from irreducible $H_{n_i}^{\Lambda_i}$-modules.

We provide a dictionary between multisegments and certain colored multipartitions, "Kleshchev multipartitions". The reverse map is obvious, but the forward one is more subtle. Only certain colored multipartitions can arise. In Section 10 we explain those colored multipartitions correspond to nodes in a connected component of a tensor product of level 1 crystals.

We can use Theorem 3.2 to identify for which $\mathbf{\Delta}$ is $M_{\mathbf{\Delta}} \in \operatorname{Rep} H_n^{\lambda}$.

Theorem 9.1. *The $\mathbf{\Delta}$ such that $M_{\mathbf{\Delta}} \in \operatorname{Rep} H_n^{\Lambda_i}$ are of the form*

$$\mathbf{\Delta} = \{\triangle_{(i,b_1)}, \triangle_{(i-1,b_2)}, \ldots, \triangle_{(i-k+1,b_k)}\},$$

where $b_1 > b_2 > \ldots > b_k$, or $\mathbf{\Delta} = \varnothing$.

In other words, we can associate to $\mathbf{\Delta}$ the i-colored partition $\mu(\mathbf{\Delta}) = (b_1 - i + 1, b_2 - i + 2, \ldots, b_k - i + k)$, pictured as

$$\begin{array}{|c|c|c|c|c|c|}
\hline
i & i+1 & \cdots & \cdots & \cdots & b_1 \\
\hline
i-1 & i & \cdots & b_2 & \multicolumn{2}{c}{} \\
\cline{1-4}
\vdots & \vdots & \vdots & \multicolumn{3}{c}{} \\
\cline{1-3}
i-k+1 & \cdots & b_k & \multicolumn{3}{c}{} \\
\cline{1-3}
\end{array}.$$

In this case, $\mathbf{\Delta}(\mu, i) = \mathbf{\Delta}$.

Proof. To be in $\operatorname{Rep} H_n^{\Lambda_i}$ means that $\widehat{\varepsilon}_i(M_{\mathbf{\Delta}}) \leqslant 1$ and $\widehat{\varepsilon}_j(M_{\mathbf{\Delta}}) = 0$ if $j \neq i$. Rule 2 for computing $\widehat{\varepsilon}_j(M_{\mathbf{\Delta}})$ shows that if the segment $\triangle_{(j,b)}$ occurs in $\mathbf{\Delta}$ and $j \neq i$, then it must be preceded in the left order by $\triangle_{(j+1,b')}$, which forces $b < b'$. Further, if $\mathbf{\Delta} \neq \varnothing$, then some segment $\triangle_{(i,b_1)}$ must occur. Only the $\mathbf{\Delta}$ listed above satisfy these requirements. $\square$

It is no surprise that the multisegments above correspond to partitions. We note that for generic q, $H_n^{\Lambda_0} \simeq RS_n$, the group algebra of the symmetric group. The irreducible modules of S_n are parameterized by partitions. The irreducible module parameterized by μ is called the Specht module S^{μ} and it is isomorphic to $M_{\mathbf{\Delta}(\mu,0)}$. (The modules for $H_n^{\Lambda_i}$ differ only in that X_1 acts by the scalar q^i.)

Recall we defined $\mathbf{\Delta}(\mu, i) = \{\triangle_{(i,i+\mu_1-1)}, \ldots, \triangle_{(i-k+1,i-k+\mu_k)}\}$, where k is the length of μ. Hence, $\mu\left(\mathbf{\Delta}(\mu, i)\right) = \mu$ undoes this operation. Theorem 3.4 explains how to associate a multipartition to an arbitrary multisegment.

Also recall that $N_{\mu[i]} = M_{\mathbf{\Delta}(\mu,i)}$, with the convention that $N_{\mathbf{0}[i]} = 0$.

Lemma 8. (1) *For all j, $\varepsilon_j(N_{\mu[i]}) \leqslant 1$.*

(2) $\widetilde{e}_j N_{\mu[i]} = N_{\widetilde{\mathbf{e}}_{\mathbf{j}}(\mu[\Lambda_i])}$, *where, as in Rule 3, $\widetilde{\mathbf{e}}_{\mathbf{j}}(\mu[\Lambda_i])$ denotes the removal of a removable j box from the diagram of μ when colored by i, or denotes $\mathbf{0}$ if no such removable j box exists.*

Proof. In the multisegment $\mathbf{\Delta} = \mathbf{\Delta}(\mu, i) = \{\triangle_{(i,b_1)}, \ldots, \triangle_{(i-k+1,b_k)}\}$ with $b_1 > \cdots > b_k$, observe each b_t is distinct. Thus in our rule for computing $\varepsilon_j(M_{\mathbf{\Delta}})$, there is at most one $-$ symbol. This shows part 1 (using Theorem 3.1).

In fact, if $\varepsilon_j(N_{\mu[i]}) = 1$, it means we must see $-$ but not $-+$ in calculating as in Rule 1. This corresponds to the i-colored diagram for μ having a removable j-box. The $-$ means some row of μ ends in a j-box. But the absence of $+$ means the rows below it must not end in a $(j-1)$-box. That means the j-box has no box directly below it (or else it would be filled with $j-1$). Then it is clearly removable, and $\mathbf{\Delta}(\mu \backslash \boxed{\text{j}}, i) = \widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}$. The converse also holds, since if $\boxed{\text{j}}$ is removable, the row beneath it cannot end in $j-1$ but nor can any of the rows beneath it since their fillings strictly decrease. Thus $\widetilde{e}_j N_{\mu[i]} = \widetilde{e}_j M_{\mathbf{\Delta}} = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\mathbf{\Delta}} = M_{\mathbf{\Delta}(\mu\backslash j,i)} = N_{\widetilde{\mathbf{e}}_{\mathbf{j}}(\mu[\Lambda_i])}$. $\square$

Theorem 9.2. (1) *Let $M_{\mathbf{\Delta}}$ be an $H_n^{\Lambda_i+\Lambda_h}$-module. Assume $i \geqslant h$. Then there exist partitions μ and ν such that*

$$M_{\mathbf{\Delta}} = \operatorname{cosoc} \operatorname{Ind} N_{\mu[i]} \boxtimes N_{\nu[h]}.$$

Further, $\mathbf{\Delta} = \mathbf{\Delta}(\mu,i) \cup \mathbf{\Delta}(\nu,h)$ and $b_{i-h+x} \leqslant c_x$, where $\mathbf{\Delta}(\mu,i) = \{\triangle_{(i,b_1)}, \ldots, \triangle_{(i-m+1,b_m)}\}$ and $\mathbf{\Delta}(\nu, h) = \{\triangle_{(h,c_1)}, \ldots, \triangle_{(h-k+1,c_k)}\}$.

(2) *Given μ, ν such that $\mu_{i-h+x} \leqslant \nu_x$ for $x \geqslant 1$,* $\operatorname{cosoc}\operatorname{Ind} N_{\mu[i]} \boxtimes N_{\nu[h]}$ *is irreducible and equal to $M_{\mathbf{\Delta}}$ where $\mathbf{\Delta} = \mathbf{\Delta}(\mu, i) \cup \mathbf{\Delta}(\nu, h)$.*

Proof. Similar to Lemma 8, $\mathbf{\Delta}$ must consist of two consecutive (with respect to the first "coordinate" of a segment) decreasing subsequences. Applying the sliding lemma will let us sort the segments comprising the subsequences into two pieces corresponding to μ and ν.

Write $\mathbf{\Delta} = \{\triangle_{(a_1,b_1)}, \ldots, \triangle_{(a_t,b_t)}\}$ in right order. If $k \neq i \neq h$, we must have $\widehat{\varepsilon}_k(M_{\mathbf{\Delta}}) = 0$ $\widehat{\varepsilon}_i(M_{\mathbf{\Delta}}) \leqslant 1$ and $\widehat{\varepsilon}_h(M_{\mathbf{\Delta}}) \leqslant 1$ (or $\widehat{\varepsilon}_i(M_{\mathbf{\Delta}}) \leqslant 2$ if $k \neq i$ and $i = h$). Theorem 3.2 shows that $\{a_1, \ldots, a_t\}$ is the union of two intervals $\{i, i-1, \ldots, i-s+1\}$ and $\{h, h-1, \ldots, h-t+s+1\}$ where their corresponding b's are also strictly decreasing.

If these intervals are disjoint, then the concatenation of the intervals is in right order and it is clear

$$\begin{aligned}
M_{\mathbf{\Delta}} &= \operatorname{cosoc}\operatorname{Ind}\mathbf{\Delta} \\
&= \operatorname{cosoc}\operatorname{Ind}\triangle_{(i,b_1)} \boxtimes \triangle_{(i-1,b_2)} \boxtimes \cdots \boxtimes \triangle_{(i-s+1,b_s)} \\
&\qquad\qquad \boxtimes \triangle_{(h,b_{s+1})} \boxtimes \cdots \boxtimes \triangle_{(h-t+s+1,b_t)} \\
&= \operatorname{cosoc}\operatorname{Ind}_{s,t}^{n}(\operatorname{cosoc}\operatorname{Ind}\triangle_{(i,b_1)} \boxtimes \cdots \boxtimes \triangle_{(i-s+1,b_s)}) \\
&\qquad\qquad \boxtimes (\operatorname{cosoc}\operatorname{Ind}\triangle_{(h,b_{s+1})} \boxtimes \cdots \boxtimes \triangle_{(h-t+s+1,b_t)}) \\
&= \operatorname{cosoc}\operatorname{Ind}_{s,t}^{n} N_{\mu[i]} \boxtimes N_{\nu[h]},
\end{aligned}$$

where $\mu = (b_1 - i + 1, \ldots, b_s - i + s)$ and $\nu = (b_{s+1} - h + 1, \ldots, b_t - h + t - s)$. By construction $\mathbf{\Delta} = \mathbf{\Delta}(\mu, i) \cup \mathbf{\Delta}(\nu, h)$.

However, there may be overlap between the intervals. In that case, we partition $\mathbf{\Delta} = \mathbf{\Delta}_1 \cup \mathbf{\Delta}_2$ as follows. If $a_m \in \{i, \ldots, i-s+1\} \setminus \{h, \ldots, h-t+s+1\}$, then put $\triangle_{(a_m,b_m)}$ in $\mathbf{\Delta}_1$. If $a_m \in \{h, \ldots, h-t+s+1\} \setminus \{i, \ldots, i-s+1\}$, then put $\triangle_{(a_m,b_m)}$ in $\mathbf{\Delta}_2$. For $a \in \{i, \ldots, i-s+1\} \cap \{h, \ldots, h-t+s+1\}$, there exist $c \geqslant b$ such that both of $\triangle_{(a,b)}$ and $\triangle_{(a,c)}$ are in the multisegment $\mathbf{\Delta}$. Put the shorter $\triangle_{(a,b)}$ in $\mathbf{\Delta}_1$ and the longer $\triangle_{(a,c)}$ in $\mathbf{\Delta}_2$. Then $\mu = \mu(\mathbf{\Delta}_1)$ and $\nu = \mu(\mathbf{\Delta}_2)$ will be partitions and $\mathbf{\Delta}_1 = \mathbf{\Delta}(\mu, i)$, $\mathbf{\Delta}_2 = \mathbf{\Delta}(\nu, h)$.

Define $\Gamma_1^a = \{\triangle_{(\alpha,\beta)} \in \mathbf{\Delta}_1 \mid \alpha > a\}$ and $\Gamma_2^a = \{\triangle_{(\alpha,\beta)} \in \mathbf{\Delta}_2 \mid \alpha > a\}$. Suppose $b \leqslant c$. Then it follows from part 2 of the sliding Lemma 6 that

$$\begin{aligned}
&\operatorname{cosoc} \operatorname{Ind} M_{\Gamma_1^a} \boxtimes M_{\Gamma_2^a} \boxtimes \triangle_{(a,c)} \boxtimes \triangle_{(a,b)} \\
&\quad = \operatorname{cosoc} \operatorname{Ind} M_{\Gamma_1^a} \boxtimes M_{\Gamma_2^a \cup \{\triangle_{(a,c)}\}} \boxtimes \triangle_{(a,b)} \\
&\quad = \operatorname{cosoc} \operatorname{Ind} M_{\Gamma_1^a} \boxtimes \triangle_{(a,b)} \boxtimes M_{\Gamma_2^a \cup \{\triangle_{(a,c)}\}} \\
&\quad = \operatorname{cosoc} \operatorname{Ind} M_{\Gamma_1^a \cup \{\triangle_{(a,b)}\}} \boxtimes M_{\Gamma_2^a \cup \{\triangle_{(a,c)}\}} \\
&\quad = \operatorname{cosoc} \operatorname{Ind} M_{\Gamma_1^{a-1}} \boxtimes M_{\Gamma_2^{a-1}}.
\end{aligned}$$

If a is such that both $\triangle_{(a,b)}, \triangle_{(a,c)} \in \mathbf{\Delta}$, then $\Gamma_1^{a-1} = \Gamma_1^a \cup \{\triangle_{(a,b)}\}$ and $\Gamma_2^{a-1} = \Gamma_2^a \cup \{\triangle_{(a,c)}\}$. Continuing as above, starting with $\mathbf{\Delta}$ in the right order, yields $M_{\mathbf{\Delta}} = \operatorname{cosoc} \operatorname{Ind} M_{\mathbf{\Delta}_1} \boxtimes M_{\mathbf{\Delta}_2} = \operatorname{cosoc} \operatorname{Ind} N_{\mu[i]} \boxtimes N_{\nu[h]}$.

Conversely, given any μ, ν with the property that $\mu_{i-h+x} \leqslant \nu_x$ for all $x \geqslant 1$ (with the convention $\mu_x = 0$ if $x > \ell(\mu)$), the above argument shows $\operatorname{cosoc} \operatorname{Ind} N_{\mu[i]} \boxtimes N_{\nu[h]}$ is irreducible and equal to $M_{\mathbf{\Delta}}$ for $\mathbf{\Delta} = \mathbf{\Delta}(\mu, i) \cup \mathbf{\Delta}(\nu, h)$.

Observe that the condition that μ and ν satisfy are equivalent to the multipartition (μ, ν) colored by $\Lambda_i + \Lambda_h$ being a Kleshchev multipartition. $\square$

Observe that $n(\mathbf{\Delta}) = |\mu| + |\nu|$. Also it is possible that μ or ν be the empty partition.

Theorem 3.4 extends the same argument to λ of any level r. Its proof is just a repetition of the proof above, now sliding the segments of $\mathbf{\Delta}$ past each other until they are sorted by the rules above into r multisegments whose first coordinate is consecutive decreasing. This sorting $\mathbf{\Delta} = \mathbf{\Delta}_1 \cup \ldots \cup \mathbf{\Delta}_r$ will correspond to a Kleshchev multipartition $\underline{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$, because the shorter of two segments with the same first coordinate is always slid to the left. We repeat the statement of the theorem here for convenience.

Theorem. (Theorem 3.4) *Let $M_{\mathbf{\Delta}}$ be an H_n^λ-module where $\lambda = \Lambda_{i_1} + \Lambda_{i_2} + \ldots + \Lambda_{i_r}$, $i_1 \geqslant \ldots \geqslant i_r$. Then there exists a Kleshchev multipartition $\underline{\mu}$ such that $M_{\mathbf{\Delta}} = N_{\underline{\mu}[\lambda]}$ and $\mathbf{\Delta} = \mathbf{\Delta}(\underline{\mu}[\lambda])$.*

Conversely, if $\underline{\mu}$ is a Kleshchev multipartition, then

$$N_{\underline{\mu}[\lambda]} = \operatorname{cosoc} \operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$$

is irreducible and isomorphic to $M_{\mathbf{\Delta}}$ for $\mathbf{\Delta} = \mathbf{\Delta}(\underline{\mu}[\lambda])$.

If $\underline{\mu}[\lambda]$ is not a Kleshchev multipartition, then $\operatorname{cosoc} \operatorname{Ind} N_{\mu^{(1)}[i_1]} \boxtimes \cdots \boxtimes N_{\mu^{(r)}[i_r]}$ need not be irreducible. Furthermore, Theorem 3.5 (repeated below) need not hold for arbitrary colored multipartitions.

The reader may recognize the condition

$$\mu^{(t)}_{i_t - i_{t+1} + x} \leqslant \mu^{(t+1)}_x$$

as the condition that the nodes in the connected component of highest weight λ satisfy in the tensor of crystal graphs $B(\Lambda_{i_r}) \otimes \cdots B(\Lambda_{i_1})$. See Section 10 and Corollary 3.7 for a description of these theorems in the language of crystal graphs.

We will now justify that the $\pm$ Rule 1 given for $\widetilde{\mathbf{e}}_{\mathbf{i}}\boldsymbol{\Delta}$ is compatible with the one for multipartitions.

Theorem. (Theorem 3.5) *Given a multisegment* $\boldsymbol{\Delta}$ *and a Kleshchev multipartition* $\underline{\mu}[\lambda]$ *such that* $\boldsymbol{\Delta} = \boldsymbol{\Delta}(\underline{\mu}[\lambda])$, *then* $\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta} = \boldsymbol{\Delta}(\widetilde{\mathbf{e}}_{\mathbf{j}}\underline{\mu}[\lambda])$. *In other words,* $\widetilde{e}_j N_{\underline{\mu}[\lambda]} = N_{\widetilde{\mathbf{e}}_{\mathbf{j}}\underline{\mu}[\lambda]}$.

Proof. When we compute $\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta}$, we require that $\boldsymbol{\Delta}$ be in right order. However, in constructing $\underline{\mu}[\lambda]$ in Theorem 3.4, that order is changed by repeated application of the sliding lemma.

We will consider all possible reorderings that occur and show they do not change the uncanceled $+$ and $-$ symbols. There are 3 cases.

Case 1. In $\boldsymbol{\Delta}$ we see $\triangle_{(a,j-1)}$ followed by $\triangle_{(a,j)}$. Then these two segments remain in that relative order. (Recall in partitioning $\boldsymbol{\Delta}$ into two pieces that our rule would put $\triangle_{(a,j-1)}$ in $\boldsymbol{\Delta}_1$ and $\triangle_{(a,j)}$ in $\boldsymbol{\Delta}_2$.)

Case 2. In $\boldsymbol{\Delta}$ we see $\triangle_{(b,j-1)}$ followed by $\triangle_{(a,j)}$ and $b > a$. Again, these two segments remain in that relative order.

Case 3. In $\boldsymbol{\Delta}$ we see $\triangle_{(b,j)}$ followed by $\triangle_{(a,j-1)}$ and $b > a$. If these two segments do switch position, it would be by applying the sliding lemma—that $\mathrm{Ind} M_\Gamma \boxtimes \triangle_{(a,j-1)} \simeq \mathrm{Ind}\triangle_{(a,j-1)} \boxtimes M_\Gamma$ where $\Gamma \ni \triangle_{(b,j)}$ and the last segment in Γ is $\triangle_{(a,c)}$ for some $j > c \geqslant j-1$. Thus, $c = j-1$. Before sliding the pattern of symbols is $- + +$, whereas afterwards it is $+ - +$. However, all adjacent $-+$ get cancelled in either rule for $\widetilde{\mathbf{e}}_{\mathbf{j}}$ and so we have not changed the uncanceled symbols (nor the segments they are attached to) at all; both configurations contribute just a $+$ attached to the $\triangle_{(a,j-1)}$.

Applying the $\pm$ rule to compute $\widetilde{e}_j M_{\boldsymbol{\Delta}} = M_{\widetilde{\mathbf{e}}_{\mathbf{j}}\boldsymbol{\Delta}}$ is thus equivalent to that of computing $\widetilde{e}_j N_{\underline{\mu}[\lambda]} = N_{\widetilde{\mathbf{e}}_{\mathbf{j}}\underline{\mu}[\lambda]}$. $\square$

Above we have built up the partitions "row by row", with each row corresponding to a segment. We could have chosen to build them "column by column" which would have meant using Steinberg modules in place of trivial modules. These are also one dimensional, but have character $(q^j q^{j-1} \cdots q^i)$, and all $T_k + 1$ vanish on them. In that case we would have a similar $+-$ rule that would correspond to looking for addable/removable boxes at the ends of columns (not rows).

10. Discussion: crystals

Now we can put the theorems of Section 3 into the language of Kashiwara's crystal graphs. We refer the reader to [K] for the definitions and properties of crystal graphs. A good reference in this context is [G], where he also includes when q is a root of unity, in which case one must be more careful.

Fix $\lambda = \Lambda_{i_1} + \ldots + \Lambda_{i_r}$ with $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_r$, a weight of the Lie algebra $\mathfrak{gl}_\infty$. Let $L(\lambda)$ denote the irreducible integrable representation of $\mathfrak{gl}_\infty$ with highest weight λ and let $B(\lambda)$ denote its crystal graph.

Let $B(\infty)$ denote the crystal graph associated to $U(\mathfrak{n}^-)$, where $\mathfrak{n}^-$ is the nilpotent subalgebra of $\mathfrak{gl}_\infty$. We can think of each node of $B(\infty)$ as labeled by a multisegment $\mathbf{\Delta}$. (Observe another interpretation of a multisegment is as a sum of positive roots of $\mathfrak{gl}_\infty$. A segment $\triangle_{(i,j)}$ corresponds to the positive root $\alpha_{ij} = \alpha_i + \ldots + \alpha_j$, and the multisegment $\mathbf{\Delta}$ corresponds to their sum.)

The following two theorems are essentially Grojnowski's Theorem 14.3.

Theorem 10.1. ([G]) *Let B_λ be the graph whose nodes are the irreducible H_n^λ-modules for $n \geqslant 0$ with edges given by $\widetilde{e}_i M \xrightarrow{i} M$ if $\widetilde{e}_i M \neq 0$. Then $B_\lambda = B(\lambda)$.*

Theorem 10.2. *Let B_{aff} be the graph whose nodes are the irreducible modules in* Rep_q *with edges given by $\widetilde{e}_i M \xrightarrow{i} M$ if $\widetilde{e}_i M \neq 0$. Then $B_{\mathrm{aff}} = B(\infty)$.*

In this language, Theorems 3.1, 3.4, and 3.5 become

Corollary 10.3. *$B_{BZ} = B_{\mathrm{aff}}$ with crystal isomorphism $\mathbf{\Delta} \mapsto M_{\mathbf{\Delta}}$.*
$B'_\lambda = B_\lambda$ with crystal isomorphism $\underline{\mu}[\lambda] \mapsto N_{\underline{\mu}[\lambda]}$.
Furthermore, $\underline{\mu}[\lambda] \mapsto \mathbf{\Delta}(\underline{\mu}[\lambda])$ coincides with the inclusion of B_λ into B_{aff}.

We can view B_λ as a subgraph of B_{aff} (or $B(\lambda) \subseteq B(\infty)$). Indeed the irreducible H_n^λ-modules are just a subset of the irreducibles of $\mathrm{Rep}_q H_n$. Rule 2 for $\widehat{\varepsilon}_i(M_{\mathbf{\Delta}})$ tells us exactly which subgraph B_λ is, i.e., which multisegments comprise it, by requiring $\widehat{\varepsilon}_i(M_{\mathbf{\Delta}})$ be less than or equal to the multiplicity of Λ_i in λ.

On the other hand, $B(\lambda) \hookrightarrow B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$. The nodes in the graph $B(\Lambda_i)$ correspond to *all* partitions μ, colored by i. (In the case $i = 0$, $L(\Lambda_0)$ is the basic representation and the crystal graph $B(\Lambda_0)$ can be identified with Young's lattice of partitions. For other $i \in \mathbb{Z}$ simply shift the edge labels by $+i$.) Thus, in the tensor product $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$, it is natural to label nodes by colored multipartitions $\underline{\mu}[\lambda]$. The nodes run over all λ-colored multipartitions, but this graph is not connected.

The connected component in $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$ of $\varnothing \otimes \varnothing \otimes \cdots \otimes \varnothing$, the unique node with weight λ, defines $B(\lambda)$ as a subgraph. These realizations of $B(\lambda)$ as a subgraph of two different crystals gives us a map from multisegments (those with $\widehat{\mathbf{e}}_{\mathbf{i}}$ bounded by λ) to colored multipartitions

$$\mathbf{\Delta} \mapsto \mu(\mathbf{\Delta}) = (\mu^{(1)}, \ldots, \mu^{(r)}) \leftrightarrow \mu^{(r)} \otimes \cdots \otimes \mu^{(1)}$$

which respects the action of $\widetilde{e}_i$.

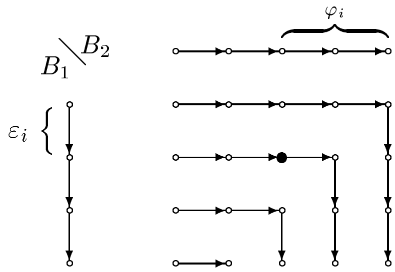
As r goes to ∞ the domain ranges over all multisegments. However we never get all colored multipartitions in the image. Theorem 3.4 explains which ones we do get, and Corollary 3.7 tells us how to interpret those in terms of a tensor product of crystal graphs.

There is a small technical point here which we must address. While multipartitions are read left to right, tensor products are read right to left. The conventional definition for tensoring crystals and the conventional definition for partitions requires that we introduce this reversal in order to be consistent.

We recall the definition of $B_2 \otimes B_1$, where B_k is a crystal graph. Its nodes are all of the form $b_2 \otimes b_1$ where b_k is a node of B_k. And so $\widetilde{e}_i$ acts via

$$\widetilde{e}_i(b_2 \otimes b_1) = \begin{cases} \widetilde{e}_i b_2 \otimes b_1 & \text{if } \varphi_i(b_2) \geqslant \varepsilon_i(b_1), \\ b_2 \otimes \widetilde{e}_i b_1 & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1). \end{cases} \tag{17}$$

This is compatible with the following picture (in which all edges $\to$ are assumed labeled by i).



The solid dot is $b_2 \otimes b_1 \in B_2 \otimes B_1$, and the rule above tells you if an arrow leading to it approaches from the top or the left.

If the B_k are arbitrary crystals, then in general $B_2 \otimes B_1$ is not isomorphic to $B_1 \otimes B_2$. However, in this paper the crystals $B(\lambda)$ correspond to integrable highest weight representations of $\mathfrak{gl}_\infty$, and so it is always the case $B(\lambda) \otimes B(\rho)$ is isomorphic to $B(\rho) \otimes B(\lambda)$. We identify the nodes of $B(\Lambda_i)$ with i-colored partitions, and in that way identify the nodes of $B(\sum_i \Lambda_i) \subseteq \bigotimes_i B(\Lambda_i)$ with multipartitions. However, the multipartitions that label the nodes of $B(\Lambda_i) \otimes B(\Lambda_j)$ can label *different* nodes of $B(\Lambda_j) \otimes B(\Lambda_i)$. (In other words, the isomorphism is *not* given by $a \otimes b \mapsto b \otimes a$.) Consequently, we are very careful of the order.

Compare the definition for $B_2 \otimes B_1$ to Theorems 3.1 and 3.5, which imply that for very special A and B,

$$\widetilde{e}_i(\operatorname{cosoc} \operatorname{Ind} A \boxtimes B) = \begin{cases} \operatorname{cosoc}(\operatorname{Ind} A \boxtimes \widetilde{e}_i B) & \text{if } \varphi_i(B) \geqslant \varepsilon_i(A), \\ \operatorname{cosoc}(\operatorname{Ind} \widetilde{e}_i A \boxtimes B) & \text{if } \varphi_i(B) < \varepsilon_i(A), \end{cases} \tag{18}$$

and also that the cosocles above are irreducible.

We will now explain (18) in more detail and in the process prove Corollary 3.7.

First we will define φ_i. We refer the reader to [G] to learn more about φ_i, which is more subtle than ε_i. It depends on the crystal operator $\widetilde{f}_i$, mentioned only briefly here and used to simplify the proof of Corollary 10.4 below.

The crystal operator $\widetilde{f}_i$ satisfies

$$\widetilde{e}_i M = N \iff M = \widetilde{f}_i N.$$

Module-theoretically, $\widetilde{f}_i N = \operatorname{cosoc} \operatorname{Ind}_{n,1}^{n+1} N \boxtimes q^i = \operatorname{cosoc} \operatorname{Ind} N \boxtimes \triangle_{(i,i)}$. This gives a recipe of how to construct an irreducible H_n-module M using the crystal graph B_{aff}, by following any path on the crystal from the node corresponding to M back to the root, one step at a time. In other words, given $\mathbf{1} = \widetilde{e}_{j_n} \cdots \widetilde{e}_{j_1} M$, we have $M = \widetilde{f}_{j_1} \cdots \widetilde{f}_{j_n} \mathbf{1}$. As in Remark 6, [BZ, Z] already gave a distinguished such path. For more on the properties of $\widetilde{f}_i$ see [G].

For B_λ, $\varphi_i(M) = \max\{m \geqslant 0 \mid \widetilde{f}_i^m M \in \operatorname{Rep} H_n^\lambda\}$. For B_{aff} it is slightly different. However, in this context, φ_i is computed by counting uncanceled $+$ symbols as in Rule 1 or 3, just as ε_i counts uncanceled $-$ symbols.

If we decree that φ_i counts uncanceled $+$ symbols, then we can see when Theorems 3.1 and 3.5 imply (18), as follows.

First, the A and B we consider are of the form $A = M_\Gamma$ and $B = M_{\Gamma'}$, where now Γ is an initial segment of $\mathbf{\Delta}$ and Γ' is a final segment, with respect to right order, and $\Gamma \cup \Gamma' = \mathbf{\Delta}$. (Note the different use of the words initial "segment" here is as initial subword with respect to right order.) Then $\mathrm{Ind}\mathbf{\Delta} = \mathrm{Ind}\Gamma \boxtimes \Gamma'$. By transitivity of induction and Remark 7, $M_{\mathbf{\Delta}} = \mathrm{cosoc}\,\mathrm{Ind} M_\Gamma \boxtimes M_{\Gamma'}$. Theorem 3.1 says that the $\pm$ rule works for all three of $M_{\mathbf{\Delta}}, M_\Gamma$, and $M_{\Gamma'}$. The $\pm$ word for $\mathbf{\Delta}$ is the concatenation of the $\pm$ words for Γ and Γ'.

$$+ + \cdots + \underbrace{- - \cdots -}_{\varepsilon_i} \quad \underbrace{+ + \cdots +}_{\varphi_i} - - \cdots -$$

If one then performs the requisite cancelling of $-+$ pairs, the leftmost uncanceled $-$ for Γ (which has a total of $\varepsilon_i(M_\Gamma)$ as yet uncanceled $-$) will remain such in $\mathbf{\Delta}$ only if it is not followed by more than $\varepsilon_i(M_\Gamma)$ many as yet uncanceled $+$, i.e., if $\varepsilon_i(M_\Gamma) > \varphi_i(M_{\Gamma'})$. But in the case $\varepsilon_i(M_\Gamma) \leqslant \varphi_i(M_{\Gamma'})$, all $-$ from Γ will get cancelled by subsequent $+$, and it is now the leftmost uncanceled $-$ from $M_{\Gamma'}$ that becomes the leftmost uncanceled $-$ for $\mathbf{\Delta}$. On the one hand, we are describing the rule for $\widetilde{e}_i M_{\mathbf{\Delta}}$, and on the other hand, we are describing (18). Notice at each stage, that all cosocles we take are irreducible (since the conditions of (18) ensure the concatenation $\widetilde{\mathbf{e}}_{\mathbf{i}}(\Gamma) \cup \Gamma'$ or $\Gamma \cup \widetilde{\mathbf{e}}_{\mathbf{i}}(\Gamma')$ is in right order).

Next, the argument for A and B of the form $N_{\underline{\mu}[\lambda]}$ is similar; but again, the multipartitions must be initial and final segments of a given multipartition, and all three must be Kleshchev multipartitions, i.e., colored multipartitions satisfying condition (4).

Consequently, we also get the following corollary, which describes how to build irreducible H_n^λ-modules out of H_a^α and H_b^β-modules when $\alpha + \beta = \lambda$.

Corollary 10.4. *Suppose α and β are weights with $\alpha + \beta = \lambda$ and $\alpha = \sum_{k=1}^{a} \Lambda_{i_k}$, $\beta = \sum_{k=a+1}^{r} \Lambda_{i_k}$, where $i_1 \geqslant i_2 \geq \ldots \geq i_r$. Suppose $\underline{\nu} \otimes \underline{\mu} \in B(\beta) \otimes B(\alpha)$ is in the connected component of the unique node with weight λ. Write $\underline{\mu} \cdot \underline{\nu} = (\mu^{(1)}, \ldots, \mu^{(a)}, \nu^{(1)}, \ldots, \nu^{(r-a)})$. Then the module*

$$N_{\underline{\mu}\cdot\underline{\nu}[\lambda]} := \mathrm{cosoc}\,\mathrm{Ind} N_{\underline{\mu}[\alpha]} \boxtimes N_{\underline{\nu}[\beta]}$$

is irreducible and

$$\widetilde{e}_i(N_{\underline{\mu}\cdot\underline{\nu}[\lambda]}) = \begin{cases} N_{\underline{\mu}\cdot\widetilde{\mathbf{e}}_{\mathbf{i}}(\underline{\nu})[\lambda]} & \text{if } \varphi_i(N_{\underline{\nu}[\beta]}) \geqslant \varepsilon_i(N_{\underline{\mu}[\alpha]}), \\ N_{\widetilde{\mathbf{e}}_{\mathbf{i}}(\underline{\mu})\cdot\underline{\nu}[\lambda]} & \text{if } \varphi_i(N_{\underline{\nu}[\beta]}) < \varepsilon_i(N_{\underline{\mu}[\alpha]}). \end{cases}$$

Proof. The proof follows from the discussion after equation (18). If we take the $\pm$ word corresponding to $\underline{\mu} \cdot \underline{\nu}[\lambda]$ but then divide that word according to how $\alpha + \beta = \lambda$, the rule for computing $\widetilde{\mathbf{e}}_{\mathbf{i}}$ coincides with the rule for computing $\widetilde{\mathbf{e}}_{\mathbf{i}}(\underline{\nu} \otimes \underline{\mu})$ in $B(\beta) \otimes B(\alpha)$.

We know by Theorem 3.5 that some sequence of $\widetilde{e}_i$ will bring $N_{\underline{\mu}\cdot\underline{\nu}[\lambda]}$ to $\mathbf{1} = N_{(\varnothing,\ldots,\varnothing)[\lambda]}$ if $\underline{\mu} \cdot \underline{\nu}$ satisfies condition (4). The argument above shows the same sequence of $\widetilde{e}_i$ will bring $\underline{\nu} \otimes \underline{\mu}$ to $\varnothing \otimes \varnothing$. Thus Kleshchev multipartitions correspond to nodes in the connected component of $\varnothing \otimes \varnothing$.

To see that every node in that connected component corresponds to some Kleshchev multipartition, we need to verify that if $\widetilde{e}_i(\underline{\nu}' \otimes \underline{\mu}') = \underline{\nu} \otimes \underline{\mu}$ and $\underline{\mu} \cdot \underline{\nu}[\lambda]$ is Kleshchev, then so is $\underline{\mu}' \cdot \underline{\nu}'[\lambda]$.

This is less cumbersome to state if we make use of the crystal operator $\widetilde{f}_i$. We define $\widetilde{\mathbf{f}}_\mathbf{i}(\underline{\mu}[\lambda])$ to add an i-box to the partition corresponding to the rightmost uncanceled $+$, if it exists, and otherwise it is $\mathbf{0}$. Thus $\widetilde{\mathbf{e}}_\mathbf{i}(\underline{\mu}'[\lambda]) = \underline{\mu}[\lambda] \iff \underline{\mu}'[\lambda] = \widetilde{\mathbf{f}}_\mathbf{i}(\underline{\mu}[\lambda])$. The argument about $\pm$ words above already shows that $\widetilde{\mathbf{f}}_\mathbf{i}(\underline{\mu} \cdot \underline{\nu}[\lambda]) = \underline{\mu}' \cdot \underline{\nu}'[\lambda]$ if $\widetilde{f}_i(\underline{\nu} \otimes \underline{\mu}) = \underline{\nu}' \otimes \underline{\mu}'$, where

$$\widetilde{f}_i(b_2 \otimes b_1) = \begin{cases} b_2 \otimes \widetilde{f}_i b_1 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ \widetilde{f}_i b_2 \otimes b_1 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases} \tag{19}$$

This definition is also compatible with the picture below (17), and "undoes" $\widetilde{e}_i$.

What we need to show is that if $\underline{\mu} \cdot \underline{\nu}[\lambda]$ is Kleshchev, then so is $\widetilde{\mathbf{f}}_\mathbf{i}(\underline{\mu} \cdot \underline{\nu}[\lambda])$, if it is nonzero. In fact, given the comment above, all we need show is that $\underline{\mu}[\lambda]$ is Kleshchev $\iff \widetilde{\mathbf{f}}_\mathbf{i}(\underline{\mu}[\lambda])$ is.

It is quite tedious to check this directly. Theorem 3.4 lets us show this indirectly. The theorem maps a multisegment to a *Kleshchev* multipartition, and Theorem 3.5 says this map intertwines the operator $\widetilde{\mathbf{e}}_\mathbf{i}$. Hence it also intertwines $\widetilde{\mathbf{f}}_\mathbf{i}$ (except possibly when $\widetilde{\mathbf{f}}_\mathbf{i}(\underline{\mu}[\lambda]) = \mathbf{0}$, but in this case there is nothing to check). Hence $\widetilde{\mathbf{f}}_\mathbf{i}(\underline{\mu}[\lambda])$ must be Kleshchev since it comes from some multisegment. $\square$

We point out that when A and B are arbitrary modules, we need not have the module $\operatorname{cosoc}\operatorname{Ind} A \boxtimes B$ be irreducible, nor have (18) hold. Furthermore, although $B(\lambda)$ is a subgraph of $B(\beta) \otimes B(\alpha)$ if $\alpha + \beta = \lambda$, there is no reason to expect (18) should hold for arbitrary $A \in \operatorname{Rep} H_a^\alpha$, $B \in \operatorname{Rep} H_b^\beta$. We are only certain it holds when α is an initial segment of λ and β a final segment (in the sense that λ corresponds to $i_1 i_2 \cdots i_r$, with $i_1 \geqslant \ldots \geqslant i_r$) and when $B \otimes A$ corresponds to a node in the connected component with highest weight λ. We also point out that the ability to order segments or partitions, as we do, relies on q being generic.

The argument above for $\mathbf{\Delta} = \Gamma \cup \Gamma'$ (or $\lambda = \alpha + \beta$) also works if we chop a $\pm$ word for a Kleshchev multipartition into r pieces. (The case $r = 2$ is above in Corollary 10.4.) Then Rule 3 for computing $\widetilde{\mathbf{e}}_\mathbf{i}(\underline{\mu}[\lambda])$ is exactly the rule for computing the crystal operator $\widetilde{e}_i$ on the element $\mu^{(r)} \otimes \cdots \otimes \mu^{(2)} \otimes \mu^{(1)} \in B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_2}) \otimes B(\Lambda_{i_1})$.

The reason we are in the connected component of $\varnothing \otimes \cdots \otimes \varnothing$ is the same as above. This then sketches the proof of Corollary 3.7, that the inclusion $\underline{\mu}[\lambda] \mapsto \underline{\mu}[\lambda]$ of Kleshchev multipartitions into *all* λ-colored multipartitions coincides with the inclusion of $B(\lambda)$ in $B(\Lambda_{i_r}) \otimes \cdots \otimes B(\Lambda_{i_1})$.

In fact, the "true" definition of Kleshchev multipartitions, and one that extends to q a root of unity, is that they are colored multipartitions corresponding to nodes in that connected component. For a condition parallel to (4) see [JMMO].

Remark 8. We make a final comment about the right/left disparity encountered. The "right to left" tensoring of crystals *is* compatible with computing $\widehat{\mathbf{e}}_\mathbf{i}$ and left order.

Compare equation (17) to

$$\widehat{e}_i(\operatorname{cosoc}\operatorname{Ind} M_\Gamma \boxtimes M_{\Gamma'}) = \begin{cases} \operatorname{cosoc}(\operatorname{Ind}\widehat{e}_i M_\Gamma \boxtimes M_{\Gamma'}) & \text{if } \widehat{\varphi}_i(M_\Gamma) \geqslant \widehat{\varepsilon}_i(M_{\Gamma'}), \\ \operatorname{cosoc}(\operatorname{Ind} M_\Gamma \boxtimes \widehat{e}_i M_{\Gamma'}) & \text{if } \widehat{\varphi}_i(M_\Gamma) < \widehat{\varepsilon}_i(M_{\Gamma'}), \end{cases}$$

which happens in the case Γ is an initial segment of $\underline{\boldsymbol{\Delta}}$ and Γ' is a final segment, with respect to left order, and $\Gamma \cup \Gamma' = \underline{\boldsymbol{\Delta}}$.

Just as $\widehat{\varepsilon}_i$ counted the number of uncanceled $-$ signs when computing $\widehat{\mathbf{e}}_{\mathbf{i}}$ in Rule 2, $\widehat{\varphi}_i$ counts the number of uncanceled $+$ signs.

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