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Tableaux on Periodic Skew Diagrams and Irreducible Representations of the Double Affine Hecke Algebra of Type A

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1 Introduction

As is well known, Young diagrams consisting of n boxes parameterize isomorphism classes of finite-dimensional irreducible representations of the symmetric group $\mathfrak{S}_n$ of degree n, and moreover the structure of each irreducible representation is described in terms of tableaux on the corresponding Young diagram; namely, a basis of the representation is labeled by standard tableaux, with which the action of $\mathfrak{S}_n$ generators is explicitly described. This combinatorial description due to A. Young has played an essential role in the study of the representation theory of the symmetric group (or the affine Hecke algebra), and its generalization to the (degenerate) affine Hecke algebra $H_n(q)$ of GL_n has been given in [3, 8, 9], where skew Young diagrams appear on combinatorial side.

The purpose of this paper is to introduce an "affine analogue" of skew Young diagrams and tableaux, which give a parameterization and a combinatorial description of a family of irreducible representations of the double affine Hecke algebra $\ddot{H}_n(q)$ of GL_n over a field $\mathbb{F}$, where $q \in \mathbb{F}$ is a parameter of the algebra.

The double affine Hecke algebra was introduced by Cherednik [2, 4] and has since been used by him and by several authors to obtain important results about diagonal coinvariants, Macdonald polynomials, and certain Macdonald identities.

In this paper, we focus on the case where q is not a root of 1, and we consider representations of $\ddot{H}_n(q)$ that are $\mathfrak{X}$-semisimple; namely, we consider representations which

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have a basis of simultaneous eigenvectors with respect to all elements in the commutative subalgebra $\mathbb{F}[\mathfrak{X}] = \mathbb{F}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, \xi^{\pm 1}]$ of $\ddot{H}_n(q)$. (In [8, 9], such representations for affine Hecke algebras are referred to as "calibrated.")

On combinatorial side, we introduce *periodic skew diagrams* as skew Young diagrams consisting of infinitely many boxes satisfying certain periodicity conditions. We define a tableau on a periodic skew diagram as a bijection from the diagram to $\mathbb{Z}$ which satisfies the condition reflecting the periodicity of the diagram.

Periodic skew diagrams are natural generalization of skew Young diagrams and have appeared in [5] (or implicitly in [1]), but the notion of tableaux on them seems new.

To connect the combinatorics with the representation theory of the double affine Hecke algebra $\ddot{H}_n(q)$, we construct, for each periodic skew diagram, an $\ddot{H}_n(q)$-module that has a basis of $\mathbb{F}[\mathfrak{X}]$-weight vectors labeled by standard tableaux on the diagram by giving the explicit action of the $\ddot{H}_n(q)$ generators.

Such modules are $\mathfrak{X}$-semisimple by definition. We show that they are irreducible, and that our construction gives a one-to-one correspondence between the set of periodic skew diagrams and the set of isomorphism classes of irreducible representations of the double affine Hecke algebra that are $\mathfrak{X}$-semisimple.

The classification results here recover those of Cherednik's in [5] (see also [6]), but, in this paper, we provide a detailed proof based on purely combinatorial arguments concerning standard tableaux on periodic skew diagrams.

Note that the corresponding results for the degenerate double affine Hecke algebra of GL_n easily follow from a parallel argument.

An outline of the paper is as follows. Section 2 is a review of the affine root system and the extended affine Weyl group of $\widehat{\mathfrak{gl}}_n$.

The contents of Section 3 are purely combinatorial. We introduce periodic skew diagrams and tableaux on them in Sections 3.1 and 3.2, respectively. These combinatorial objects are considered worth studying in themselves, and here we investigate their relation with the affine Weyl group and *content* functions. The set of tableaux on a periodic skew diagram admits an action of the extended affine Weyl group $\dot{W}$, and it turns out that this action is simply transitive and gives a bijective correspondence between the tableaux and the elements of $\dot{W}$. In Section 3.5, we explicitly describe the subset $\dot{W}$ corresponding to the set of the standard tableaux, which is the most interesting class from the view point of the representation theory.

We study content functions, in particular, those associated with standard tableaux, in Section 3.6. The results obtained here lay the foundation to show our construction exhausts all $\mathfrak{X}$-semisimple irreducible modules.

In Section 4, we introduce the double affine Hecke algebra and apply the combinatorics studied in Section 3 to its representation theory.

We remind the reader in Section 4.1 of the definition of the algebra $\ddot{H}_n(q)$ and review intertwining operators, which were also introduced by Cherednik and are elementary tools in the representation theory of $\ddot{H}_n(q)$.

We derive some of rigid properties of $\mathfrak{X}$-semisimple modules in Section 4.2. Then we give a combinatorial and explicit construction of the representations of $\ddot{H}_n(q)$ in Section 4.3 using tableaux on periodic skew diagrams. The related combinatorics is similar to and inspired by that in [9] for the affine Hecke algebra.

Note that the statements of Section 4.2 also hold in the case that q is a root of unity. However, when q is a root of unity, the combinatorial description of the modules is incredibly complicated.

It is proved in Section 4.4 that we have constructed all the $\mathfrak{X}$-semisimple irreducible representations and that they are distinct up to diagonal shift of periodic skew diagrams. This gives the classification of the $\mathfrak{X}$-semisimple irreducible representations of $\ddot{H}_n(q)$.

2 The affine root system and Weyl group

Let $\mathbb{Q}$ denote the field of rational numbers, and let $\mathbb{Z}$ denote the ring of integral numbers. We use the notation

$$\mathbb{Z}_{\geq k} = \{n \in \mathbb{Z} \mid n \geq k\} \tag{2.1}$$

for $k \in \mathbb{Z}$, and

$$[i, j] = \{i, i+1, \dots, j\} \tag{2.2}$$

for $i, j \in \mathbb{Z}$ with $i \leq j$.

2.1 The affine root system

Let $n \in \mathbb{Z}_{\geq 2}$. Let $\tilde{\mathfrak{h}}$ be an $(n+2)$-dimensional vector space over $\mathbb{Q}$ with the basis $\{\epsilon_1^\vee, \epsilon_2^\vee, \dots, \epsilon_n^\vee, c, d\}$:

$$\tilde{\mathfrak{h}} = \left(\bigoplus_{i=1}^{n} \mathbb{Q}\epsilon_i^\vee\right) \oplus \mathbb{Q}c \oplus \mathbb{Q}d. \tag{2.3}$$

Introduce the nondegenerate symmetric bilinear form $(|)$ on $\tilde{\mathfrak{h}}$ by

$$\begin{aligned}&\left(\epsilon_i^\vee|\epsilon_j^\vee\right)=\delta_{ij}, \qquad \left(\epsilon_i^\vee|c\right)=\left(\epsilon_i^\vee|d\right)=0,\\&(c|d)=1, \qquad (c|c)=(d|d)=0.\end{aligned} \tag{2.4}$$

Put $\mathfrak{h}=\bigoplus_{i=1}^n \mathbb{Q}\epsilon_i^\vee$ and $\dot{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{Q}c$.

Let $\tilde{\mathfrak{h}}^*=(\bigoplus_{i=1}^n \mathbb{Q}\epsilon_i)\oplus\mathbb{Q}c^*\oplus\mathbb{Q}\delta$ be the dual space of $\tilde{\mathfrak{h}}$, where ϵ_i, c^*, and δ are the dual vectors of $\epsilon_i^\vee$, c, and d, respectively.

We identify the dual space $\dot{\mathfrak{h}}^*$ of $\dot{\mathfrak{h}}$ as a subspace of $\tilde{\mathfrak{h}}^*$ via the identification $\dot{\mathfrak{h}}^*=\tilde{\mathfrak{h}}^*/\mathbb{Q}\delta\cong\mathfrak{h}^*\oplus\mathbb{Q}c^*$.

The natural pairing is denoted by $\langle|\rangle:\tilde{\mathfrak{h}}^*\times\tilde{\mathfrak{h}}\to\mathbb{Q}$. There exists an isomorphism $\tilde{\mathfrak{h}}^*\to\tilde{\mathfrak{h}}$ such that $\epsilon_i\mapsto\epsilon_i^\vee$, $\delta\mapsto c$, and $c^*\mapsto d$. We denote by $\zeta^\vee\in\tilde{\mathfrak{h}}$ the image of $\zeta\in\tilde{\mathfrak{h}}^*$ under this isomorphism. Introduce the bilinear form $(|)$ on $\tilde{\mathfrak{h}}^*$ through this isomorphism. Note that

$$(\zeta\mid\eta)=\left\langle\zeta\mid\eta^\vee\right\rangle=\left(\zeta^\vee\mid\eta^\vee\right)\quad\left(\zeta,\eta\in\tilde{\mathfrak{h}}^*\right). \tag{2.5}$$

Put $\alpha_{ij}=\epsilon_i-\epsilon_j$ $(1\le i\ne j\le n)$ and $\alpha_i=\alpha_{i\,i+1}$ $(1\le i\le n-1)$. Then

$$\begin{aligned}&R=\left\{\alpha_{ij}\mid i,\ j\in[1,n],\ i\ne j\right\}, \qquad R^+=\left\{\alpha_{ij}\mid i,\ j\in[1,n],\ i<j\right\},\\&\Pi=\left\{\alpha_1,\alpha_2,\dots,\alpha_{n-1}\right\}\end{aligned} \tag{2.6}$$

give the system of roots, positive roots, and simple roots of type A_{n-1}, respectively.

Put $\alpha_0=-\alpha_{1n}+\delta$, and define the set $\dot{R}$ of (real) roots, $\dot{R}^+$ of positive roots, and $\dot{\Pi}$ of simple roots of type $A_{n-1}^{(1)}$ by

$$\begin{aligned}&\dot{R}=\left\{\alpha+k\delta\mid\alpha\in R,\ k\in\mathbb{Z}\right\},\\&\dot{R}^+=\left\{\alpha+k\delta\mid\alpha\in R^+,\ k\in\mathbb{Z}_{\ge0}\right\}\sqcup\left\{-\alpha+k\delta\mid\alpha\in R^+,\ k\in\mathbb{Z}_{\ge1}\right\},\\&\dot{\Pi}=\left\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\right\}.\end{aligned} \tag{2.7}$$

2.2 Affine Weyl group

Definition 2.1. For $n\in\mathbb{Z}_{\ge2}$, the *extended affine Weyl group* $\dot{W}_n$ of $\mathfrak{gl}_n$ is the group defined by the following generators and relations:

(1) generators:

$$s_0,s_1,\dots,s_{n-1},\pi^{\pm1}; \tag{2.8}$$

(2) relations for $n \geq 3$:

$$\begin{aligned}
&s_i^2 = 1 \quad \big(i \in [0, n-1]\big),\\
&s_i s_j s_i = s_j s_i s_j \quad (i - j \equiv \pm 1 \bmod n),\\
&s_i s_j = s_j s_i \quad (i - j \not\equiv \pm 1 \bmod n),\\
&\pi s_i = s_{i+1}\pi \quad \big(i \in [0, n-2]\big), \qquad \pi s_{n-1} = s_0 \pi,\\
&\pi\pi^{-1} = \pi^{-1}\pi = 1;
\end{aligned} \tag{2.9}$$

(3) relations for $n = 2$:

$$\begin{aligned}
&s_0^2 = s_1^2 = 1,\\
&\pi s_0 = s_1 \pi, \qquad \pi s_1 = s_0 \pi,\\
&\pi\pi^{-1} = \pi^{-1}\pi = 1.
\end{aligned} \tag{2.10}$$

The subgroup W_n of $\dot{W}_n$ generated by the elements $s_1, s_2, \dots, s_{n-1}$ is called the *Weyl group* of $\mathfrak{gl}_n$. The group W_n is isomorphic to the symmetric group of degree n.

In the following, we fix $n \in \mathbb{Z}_{\geq 2}$ and denote $\dot{W} = \dot{W}_n$ and $W = W_n$.

Put

$$P = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i. \tag{2.11}$$

Put $\tau_{\epsilon_1} = \pi s_{n-1} \cdots s_2 s_1$ and $\tau_{\epsilon_i} = \pi^{i-1}\tau_{\epsilon_1}\pi^{-i+1}$ $(i \in [2, n])$. Then there exists a group embedding $P \to \dot{W}$ such that $\epsilon_i \mapsto \tau_{\epsilon_i}$. By τ_η we denote the element in $\dot{W}$ corresponding to $\eta \in P$. It is well known that the group $\dot{W}$ is isomorphic to the semidirect product $P \rtimes W$ with the relation $w\tau_\eta w^{-1} = \tau_{w(\eta)}$.

The group $\dot{W}$ acts on $\tilde{\mathfrak{h}}$ by

$$\begin{aligned}
&s_i(h) = h - \langle \alpha_i | h\rangle \alpha_i^\vee \quad \text{for } i \in [1, n-1],\ h \in \tilde{\mathfrak{h}},\\
&\tau_{\epsilon_i}(h) = h + \langle \delta | h\rangle \epsilon_i^\vee - \left(\langle \epsilon_i | h\rangle + \frac{1}{2}\langle \delta | h\rangle\right) c \quad \text{for } i \in [1, n],\ h \in \tilde{\mathfrak{h}}.
\end{aligned} \tag{2.12}$$

The dual action on $\tilde{\mathfrak{h}}^*$ is given by

$$\begin{aligned}
&s_i(\zeta) = \zeta - \big(\alpha_i | \zeta\big)\alpha_i \quad \text{for } i \in [1, n-1],\ \zeta \in \tilde{\mathfrak{h}}^*,\\
&\tau_{\epsilon_i}(\zeta) = \zeta + (\delta|\zeta)\epsilon_i - \left(\big(\epsilon_i | \zeta\big) + \frac{1}{2}(\delta|\zeta)\right)\delta \quad \text{for } i \in [1, n],\ h \in \tilde{\mathfrak{h}}^*.
\end{aligned} \tag{2.13}$$

With respect to these actions, the inner products on $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}^*$ are $\dot{W}$-invariant. Note that the set $\dot{R}$ of roots is preserved by the dual action of $\dot{W}$ on $\tilde{\mathfrak{h}}^*$. Note also that the action of $\dot{W}$ preserves the subspace $\dot{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{Q}c$, and the dual action of $\dot{W}$ on $\dot{\mathfrak{h}}^*$ (called the affine action) is described as follows:

$$\begin{aligned} s_i(\zeta) &= \zeta - (\alpha_i|\zeta)\alpha_i \quad \text{for } i \in [1, n-1],\ \zeta \in \dot{\mathfrak{h}}^*, \\ \tau_{\epsilon_i}(\zeta) &= \zeta + (\delta|\zeta)\epsilon_i \quad \text{for } i \in [1, n],\ h \in \dot{\mathfrak{h}}^*. \end{aligned} \tag{2.14}$$

For $\alpha \in \dot{R}$, there exists $i \in [0, n-1]$ and $w \in \dot{W}$ such that $w(\alpha_i) = \alpha$. We set $s_\alpha = w s_i w^{-1}$. Then s_α is independent of the choice of i and w, and we have

$$s_\alpha(h) = h - \langle \alpha \mid h \rangle \alpha^\vee \tag{2.15}$$

for $h \in \tilde{\mathfrak{h}}$. The element s_α is called the reflection corresponding to α. Note that $s_{\alpha_i} = s_i$. For $w \in \dot{W}$, set

$$R(w) = \dot{R}^+ \cap w^{-1}\dot{R}^-, \tag{2.16}$$

where $\dot{R}^- = \dot{R} \setminus \dot{R}^+$. The length $l(w)$ of $w \in \dot{W}$ is defined as the number $\sharp R(w)$ of elements in $R(w)$. For $w \in \dot{W}$, an expression $w = \pi^k s_{j_1} s_{j_2} \cdots s_{j_m}$ is called a reduced expression if $m = l(w)$. It can be seen that

$$R(w) = \{s_{j_m} \cdots s_{j_2}(\alpha_{j_1}), s_{j_m} \cdots s_{j_3}(\alpha_{j_2}), \dots, \alpha_{j_m}\} \tag{2.17}$$

if $w = \pi^k s_{j_1} s_{j_2} \cdots s_{j_m}$ is a reduced expression.

Define the Bruhat order $\preceq$ in $\dot{W}$ by

$$x \preceq w \iff x \text{ is equal to a subexpression of a reduced expression of } w. \tag{2.18}$$

We will review some fundamental facts in the theory of Coxeter groups, which are often used in this paper. See, for example, [7] for proofs.

Lemma 2.2. (i) Let $w \in \dot{W}$ and $i \in [0, n-1]$. Then

$$\begin{aligned} l(ws_i) > l(w) &\iff w(\alpha_i) \in \dot{R}^+, \\ l(s_i w) > l(w) &\iff w^{-1}(\alpha_i) \in \dot{R}^+. \end{aligned} \tag{2.19}$$

(ii) (Strong exchange condition.) Let $\alpha \in \dot{R}^+$ and let $w \in \dot{W}$ with a reduced expression $w = \pi^r s_{i_1} s_{i_2} \cdots s_{i_k}$. If $l(ws_\alpha) < l(w)$, then there exists $p \in [1, k]$ such that $ws_\alpha = \pi^r s_{i_1} s_{i_2} \cdots \widehat{s}_{i_p} \cdots s_{i_k}$ (omitting s_{i_p}). Further $\alpha = s_{i_k} s_{i_{k+1}} \cdots s_{i_{p+1}}(\alpha_{i_p})$. □

Let I be a subset of $[0, n-1]$. Put

$$\dot{\Pi}_I = \{\alpha_i \mid i \in I\} \subseteq \dot{\Pi},$$
$$\dot{W}_I = \langle s_i \mid i \in I \rangle \subseteq \dot{W}, \tag{2.20}$$
$$\dot{R}_I^+ = \{\alpha \in \dot{R}^+ \mid s_\alpha \in \dot{W}_I\}.$$

The subgroup $\dot{W}_I$ is called the parabolic subgroup corresponding to $\dot{\Pi}_I$. Define

$$\dot{W}^I = \{w \in \dot{W} \mid R(w) \cap \dot{R}_I^+ = \varnothing\}. \tag{2.21}$$

The following fact is well known.

Proposition 2.3. (i) $\dot{W}^I = \{w \in \dot{W} \mid l(ws_{\alpha_i}) > l(w)\ \forall \alpha_i \in \dot{\Pi}_I\}$.

(ii) For any $w \in \dot{W}$, there exist a unique $x \in \dot{W}^I$ and a unique $y \in \dot{W}_I$ such that $w = xy$. Namely, the set $\dot{W}^I$ gives a complete set of minimal length coset representatives for $\dot{W}/\dot{W}_I$. □

2.3 Notation

For any integer i, we introduce the following notation:

$$\epsilon_i = \epsilon_{\underline{i}} - k\delta \in \tilde{\mathfrak{h}}^*, \qquad \epsilon_i^\vee = \epsilon_{\underline{i}}^\vee - kc \in \tilde{\mathfrak{h}}, \tag{2.22}$$

where $i = \underline{i} + kn$ with $\underline{i} \in [1, n]$ and $k \in \mathbb{Z}$.

Put $\alpha_{ij} = \epsilon_i - \epsilon_j$ and $\alpha_{ij}^\vee = \epsilon_i^\vee - \epsilon_j^\vee$ for any $i, j \in \mathbb{Z}$. Noting that $\epsilon_0 - \epsilon_1 = \delta + \epsilon_n - \epsilon_1 = \alpha_0$, we reset $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $\alpha_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee$ for any $i \in \mathbb{Z}$.

The following is easy.

Lemma 2.4. (i) $\alpha_{i+n, j+n} = \alpha_{ij}$ for all $i, j \in \mathbb{Z}$.

(ii) $\dot{R} = \{\alpha_{ij} \mid i, j \in \mathbb{Z},\ i \not\equiv j \bmod n\}$ as a subset of $\tilde{\mathfrak{h}}^*$.

(iii) $\dot{R}^+ = \{\alpha_{ij} \mid i, j \in \mathbb{Z},\ i \not\equiv j \bmod n \text{ and } i < j\}$ as a subset of $\tilde{\mathfrak{h}}^*$. □

Define the action of $\dot{W}$ on the set $\mathbb{Z}$ of integers by

$$\begin{aligned} &s_i(j) = j + 1 \quad \text{for } j \equiv i \bmod n,\\ &s_i(j) = j - 1 \quad \text{for } j \equiv i + 1 \bmod n,\\ &s_i(j) = j \quad \text{for } j \not\equiv i,\ i+1 \bmod n,\\ &\pi(j) = j + 1 \quad \forall j. \end{aligned} \tag{2.23}$$

It is easy to see that the action of τ_{ϵ_i} $(i \in [1, n])$ is given by

$$\begin{aligned} &\tau_{\epsilon_i}(j) = j + n \quad \text{for } j \equiv i \mod n, \\ &\tau_{\epsilon_i}(j) = j \quad \text{for } j \not\equiv i \mod n, \end{aligned} \tag{2.24}$$

and that the following formula holds for any $w \in \dot{W}$:

$$w(j + n) = w(j) + n \quad \forall\, j. \tag{2.25}$$

Lemma 2.5. Let $w \in \dot{W}$.

(i) $w(\epsilon_j) = \epsilon_{w(j)}$ and $w(\epsilon_j^\vee) = \epsilon_{w(j)}^\vee$ for any $j \in \mathbb{Z}$.

(ii) $w(\alpha_{ij}) = \alpha_{w(i)w(j)}$ and $w(\alpha_{ij}^\vee) = \alpha_{w(i)w(j)}^\vee$ for any $i, j \in \mathbb{Z}$. □

Proof. (i) It is enough to check the statement when $w = s_i$ $(i \in [1, n-1])$ and when $w = \tau_{\epsilon_i}(i \in [1, n])$. Let $j = \underline{j} + kn$ with $\underline{j} \in [1, n]$ and $k \in \mathbb{Z}$. For $i \in [1, n-1]$, we have $s_i(\epsilon_j) = \epsilon_j - (\alpha_i \mid \epsilon_j)\alpha_i = \epsilon_j - (\alpha_i \mid \epsilon_{\underline{j}})\alpha_i$. This leads to $s_i(\epsilon_j) = \epsilon_{s_i(j)}$. For $i \in [1, n]$, we have $\tau_{\epsilon_i}(\epsilon_j) = \epsilon_j + (\epsilon_i \mid \epsilon_j)\delta = \epsilon_j + (\epsilon_i \mid \epsilon_{\underline{j}})\delta$. This leads to $\tau_{\epsilon_i}(\epsilon_j) = \epsilon_{\tau_{\epsilon_i}(j)}$.

(ii) The proof follows directly from (i). ■

3 Periodic skew diagrams and tableaux on them

Throughout this paper, we let $\mathbb{F}$ denote a field whose characteristic is not equal to 2.

3.1 Periodic skew diagrams

For $m \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$, put

$$\widehat{\mathcal{P}}^{+}_{m,\ell} = \{\mu \in \mathbb{Z}^m \mid \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m, \ell \geq \mu_1 - \mu_m\}, \tag{3.1}$$

where μ_i denotes the ith component of μ, that is, $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$. Fix $n \in \mathbb{Z}_{\geq 2}$ and introduce the following subsets of $\mathbb{Z}^m \times \mathbb{Z}^m$:

$$\begin{aligned} \widehat{\mathcal{J}}^{n}_{m,\ell} &= \left\{(\lambda, \mu) \in \widehat{\mathcal{P}}^{+}_{m,\ell} \times \widehat{\mathcal{P}}^{+}_{m,\ell} \middle| \lambda_i \geq \mu_i \;(i \in [1, m]), \sum_{i=1}^{m} (\lambda_i - \mu_i) = n\right\}, \\ \widehat{\mathcal{J}}^{*n}_{m,\ell} &= \left\{(\lambda, \mu) \in \widehat{\mathcal{P}}^{+}_{m,\ell} \times \widehat{\mathcal{P}}^{+}_{m,\ell} \middle| \lambda_i > \mu_i \;(i \in [1, m]), \sum_{i=1}^{m} (\lambda_i - \mu_i) = n\right\}. \end{aligned} \tag{3.2}$$

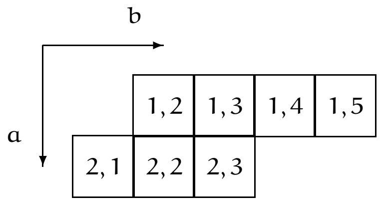

Figure 3.1

For $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, define the subsets λ/μ and $\widehat{\lambda/\mu}_{(m,-\ell)}$ of $\mathbb{Z}^2$ by

$$\begin{aligned} \lambda/\mu &= \left\{(a, b) \in \mathbb{Z}^2 \mid a \in [1, m],\ b \in [\mu_a + 1, \lambda_a]\right\}, \\ \widehat{\lambda/\mu}_{(m,-\ell)} &= \left\{(a + km, b - k\ell) \in \mathbb{Z}^2 \mid (a, b) \in \lambda/\mu,\ k \in \mathbb{Z}\right\}. \end{aligned} \tag{3.3}$$

Let $\lambda/\mu[k] = \lambda/\mu + k(m, -\ell)$. Obviously we have

$$\widehat{\lambda/\mu}_{(m,-\ell)} = \bigsqcup_{k\in\mathbb{Z}} \lambda/\mu[k] = \bigsqcup_{k\in\mathbb{Z}} \big(\lambda/\mu + k(m, -\ell)\big). \tag{3.4}$$

The set λ/μ is the so-called skew diagram (or skew Young diagram) associated with (λ, μ).

We call the set $\widehat{\lambda/\mu}_{(m,-\ell)}$ the periodic skew diagram associated with (λ, μ).

We will denote $\widehat{\lambda/\mu}_{(m,-\ell)}$ just by $\widehat{\lambda/\mu}$ when m and ℓ are fixed.

Example 3.1. (i) Let $n = 7$, $m = 2$, and $\ell = 3$. Put $\lambda = (5, 3)$, $\mu = (1, 0)$. Then $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\ell}$ and we have

$$\lambda/\mu = \left\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3)\right\}. \tag{3.5}$$

The set λ/μ is expressed by Figure 3.1 (usually, the coordinates in the boxes are omitted).

The periodic skew diagram

$$\widehat{\lambda/\mu}_{(2,-3)} = \bigsqcup_{k\in\mathbb{Z}} \lambda/\mu[k] = \bigsqcup_{k\in\mathbb{Z}} \big(\lambda/\mu + k(2, -3)\big) \tag{3.6}$$

is expressed by Figure 3.2.

Generally, periodic skew diagrams are defined as follows (see [5]).

Definition 3.2. For $\gamma \in \mathbb{Z}^2$, a subset $\Lambda \subset \mathbb{Z}^2$ is called a γ-*periodic skew diagram* (or a *periodic skew diagram* of *period* γ) if it satisfies the following conditions.

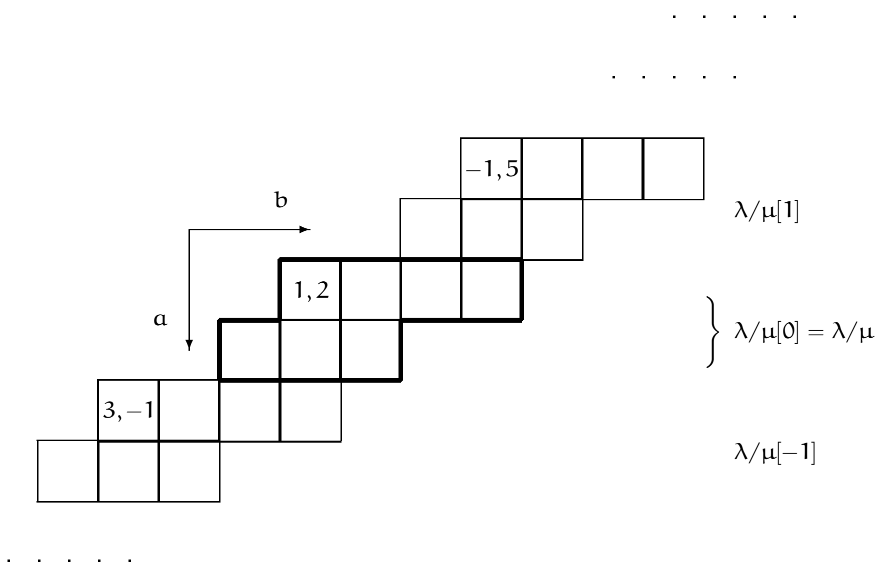

Figure 3.2

(D1) The set Λ is invariant under the parallel translation by γ:

$$\Lambda + \gamma = \Lambda, \tag{3.7}$$

and hence the group $\mathbb{Z}\gamma$ acts on Λ.

(D2) A fundamental domain of the action of $\mathbb{Z}\gamma$ on Λ consists of finitely many elements. This number is called the *degree* of Λ.

(D3) If $(a, b) \in \Lambda$ and $(a+i, b+j) \in \Lambda$ for $i, j \in \mathbb{Z}_{\geq 0}$, then the rectangle $\{(a+i', b+j') \mid i' \in [0, i], j' \in [0, j]\}$ is included in Λ.

Let $\mathcal{D}^n_\gamma$ denote the set of all γ-periodic skew diagram of degree n, and put

$$\mathcal{D}^{*n}_\gamma = \left\{\Lambda \in \mathcal{D}^n_\gamma \mid \forall\, a \in \mathbb{Z},\ \exists b \in \mathbb{Z} \text{ such that } (a, b) \in \Lambda\right\}. \tag{3.8}$$

Namely, $\mathcal{D}^{*n}_\gamma$ is the subset of $\mathcal{D}^n_\gamma$ consisting of all diagrams without empty rows.

Note that an element in $\mathcal{D}^n_{(0,0)}$ is regarded as a (classical) skew Young diagram of degree n.

Lemma 3.3. Let $\gamma \in \mathbb{Z}^2$.

(i) If $\mathcal{D}^n_\gamma \neq \varnothing$, then $\gamma \in \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0}$ or $\gamma \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0}$.

(ii) If $\mathcal{D}^{*n}_\gamma \neq \varnothing$, then $\gamma \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\leq 0}$ or $\gamma \in \mathbb{Z}_{\leq -1} \times \mathbb{Z}_{\geq 0}$. □

Proof. (i) Since $\mathcal{D}^n_\gamma = \mathcal{D}^n_{-\gamma}$, it is enough to prove that $\mathcal{D}^n_\gamma = \varnothing$ for $\gamma \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

Suppose $\mathcal{D}^n_{(m,\ell)} \neq \varnothing$ for some $m \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 1}$, and take $\Lambda \in \mathcal{D}^n_{(m,\ell)}$. Then $\bar{\Lambda} = \{(a,b) \in \Lambda \mid a \in [1,m]\}$ is a fundamental domain of the action of $\mathbb{Z}(m,\ell)$ on Λ.

Let $(a,b) \in \bar{\Lambda}$. Then condition (D1) implies $\{(a+km, b+k\ell)\}_{k\in\mathbb{Z}} \subseteq \Lambda$, and condition (D3) implies $\{(a, b+k\ell)\}_{k\in\mathbb{Z}} \subseteq \Lambda$, and hence $\{(a, b+k\ell)\}_{k\in\mathbb{Z}} \subseteq \bar{\Lambda}$. This implies that the fundamental domain $\bar{\Lambda}$ contains infinitely many elements. This contradicts condition (D2), and hence we have $\mathcal{D}^n_{(m,\ell)} = \varnothing$ for $m \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 1}$.

(ii) By (i), it is enough to show that $\mathcal{D}^{*n}_{(0,\ell)} = \varnothing$ for all $\ell \in \mathbb{Z}$, and this is easy. ■

Let $m \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$. For $(\lambda,\mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, it is easy to see that the set $\widehat{\lambda/\mu}_{(m,-\ell)}$ satisfies conditions (D1), (D2), and (D3) in Definition 3.2, and hence we have $\widehat{\lambda/\mu}_{(m,-\ell)} \in \mathcal{D}^n_{(m,-\ell)}$.

Proposition 3.4. Let $n \in \mathbb{Z}_{\geq 2}$, $m \in \mathbb{Z}_{\geq 1}$, and $\ell \in \mathbb{Z}_{\geq 0}$. The correspondence $\widehat{\mathcal{J}}^n_{m,\ell} \to \mathcal{D}^n_{(m,-\ell)}$ given by $(\lambda,\mu) \mapsto \widehat{\lambda/\mu}$ is a surjection. Moreover, its restriction to $\widehat{\mathcal{J}}^{*n}_{m,\ell}$ gives a bijection

$$\widehat{\mathcal{J}}^{*n}_{m,\ell} \xrightarrow{\sim} \mathcal{D}^{*n}_{(m,-\ell)}. \tag{3.9}$$

□

Proof. Take any $\Lambda \in \mathcal{D}^n_{(m,-\ell)}$.

Fix $i_0 \in \mathbb{Z}_{\leq 0}$ such that the i_0th row of Λ is not empty. For $i \geq i_0$, define λ_i and μ_i recursively by the following relations:

$$\begin{aligned} \lambda_i &= \begin{cases} \max\{b \in \mathbb{Z} \mid (i,b) \in \Lambda\} & \text{if the } i\text{th row is not empty,} \\ \lambda_{i-1} & \text{if the } i\text{th row is empty,} \end{cases} \\ \mu_i &= \begin{cases} \min\{b \in \mathbb{Z} \mid (i,b) \in \Lambda\} - 1 & \text{if the } i\text{th row is not empty,} \\ \lambda_{i-1} & \text{if the } i\text{th row is empty.} \end{cases} \end{aligned} \tag{3.10}$$

Put $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$. Then it follows from condition (D3) (with $i = 0$) that

$$\{(a,b) \in \Lambda \mid a = i\} = [\mu_i + 1, \lambda_i] \quad (i \in [1,m]) \tag{3.11}$$

and hence

$$\lambda/\mu = \{(a,b) \in \Lambda \mid a \in [1,m]\}. \tag{3.12}$$

It follows from condition (D1) that λ/μ is a fundamental domain of $\mathbb{Z}(m,-\ell)$ on Λ and

$$\Lambda = \bigsqcup_{k\in\mathbb{Z}} \big(\lambda/\mu + k(m,-\ell)\big) = \widehat{\lambda/\mu}_{(m,-\ell)}. \tag{3.13}$$

In particular, we have $\sharp\lambda/\mu = n$.

Note that $\lambda_0 = \lambda_m + \ell$ and $\mu_0 = \mu_m + \ell$ by condition (D1).

Now, condition (D3) implies that $\lambda_i \geq \lambda_{i+1}$ and $\mu_i \geq \mu_{i+1}$ for all $i \geq i_0$, in particular, for all $i \in [1, m-1]$. This yields $\lambda \in \widehat{\mathcal{P}}^+_{m,\ell}$ and $\mu \in \widehat{\mathcal{P}}^+_{m,\ell}$. Therefore, the correspondence $\widehat{\mathcal{J}}^n_{m,\ell} \to \mathcal{D}^n_{(m,-\ell)}$ is surjective.

Now, it is clear from the discussion above that the correspondence $(\lambda,\mu) \mapsto \widehat{\lambda/\mu}$ gives a bijection $\widehat{\mathcal{J}}^{*n}_{m,\ell} \to \mathcal{D}^{*n}_{(m,-\ell)}$. ■

3.2 Tableaux on periodic skew diagram

Fix $n \in \mathbb{Z}_{\geq 2}$. Recall that a bijection from a skew Young diagram, say λ/μ, of degree n to the set $[1, n]$ is called a tableau on λ/μ.

Definition 3.5. For $\gamma \in \mathbb{Z}^2$ and $\Lambda \in \mathcal{D}^n_\gamma$, a bijection $T : \Lambda \to \mathbb{Z}$ is said to be a γ-*tableau* on Λ if T satisfies

$$T(u+\gamma) = T(u) + n \quad \forall\, u \in \Lambda. \tag{3.14}$$

Let $\mathrm{Tab}_\gamma(\Lambda)$ denote the set of all γ-tableaux on Λ.

In this paper, we mostly treat periodic skew diagrams associated with $(\lambda,\mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ for some $m \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$. For $(\lambda,\mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, we always choose $(m,-\ell)$ as a period of $\widehat{\lambda/\mu}$. We use the abbreviated notation

$$\mathrm{Tab}(\widehat{\lambda/\mu}) = \mathop{\mathrm{Tab}}_{(m,-\ell)}(\widehat{\lambda/\mu}) \tag{3.15}$$

for $(\lambda,\mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, and we let a tableau on $\widehat{\lambda/\mu}$ mean an $(m,-\ell)$-tableau on $\widehat{\lambda/\mu}$.

Remark 3.6. A tableau on $\widehat{\lambda/\mu}$ is determined uniquely from the values on a fundamental domain of $\widehat{\lambda/\mu}$ with respect to the action of $\mathbb{Z}\gamma$. It also holds that any bijection from a fundamental domain of $\mathbb{Z}\gamma$ to the set $[1, n]$ uniquely extends to a tableau on $\widehat{\lambda/\mu}$.

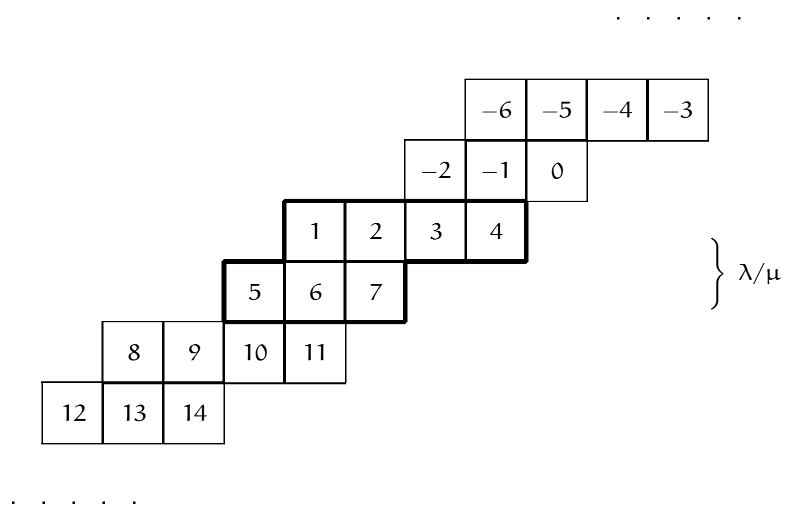

Figure 3.3

There exists a unique tableau $T_0^{\widehat{\lambda/\mu}} = T_0$ on $\widehat{\lambda/\mu}$ such that

$$T_0\big(i, \mu_i + j\big) = \sum_{k=1}^{i-1} \big(\lambda_k - \mu_k\big) + j \quad \text{for } i \in [1, m],\ j \in \big[1, \lambda_i - \mu_i\big]. \tag{3.16}$$

We call T_0 the *row reading tableau* on $\widehat{\lambda/\mu}$.

Example 3.7. Let $n = 7$, $m = 2$, $\ell = 3$ and $\lambda = (5, 3)$, $\mu = (1, 0)$. The tableau T_0 on $\widehat{\lambda/\mu}$ given above is pictured in Figure 3.3.

Proposition 3.8. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. The group $\dot{W}$ acts on the set $\mathrm{Tab}(\widehat{\lambda/\mu})$ by

$$(wT)(u) = w\big(T(u)\big) \tag{3.17}$$

for $w \in \dot{W}$, $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, and $u \in \widehat{\lambda/\mu}$. □

Proof. It is obvious that wT is a bijection. It is enough to verify that wT satisfies condition (3.14) in Definition 3.5. Putting $\gamma = (m, -\ell)$, we have

$$(wT)(u + \gamma) = w\big(T(u + \gamma)\big) = w\big(T(u) + n\big) = wT(u) + n. \tag{3.18}$$

Therefore, wT satisfies (3.14). ■

For each $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, define the map

$$\psi_T : \dot{W} \longrightarrow \mathrm{Tab}(\widehat{\lambda/\mu}) \tag{3.19}$$

by $\psi_T(w) = wT$ $(w \in \dot{W})$.

Proposition 3.9. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. For any $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, the correspondence ψ_T is a bijection. □

Proof. It is enough to show the statement for $T = T_0$ given by (3.16). We prove the surjectivity first. Take any $S \in \mathrm{Tab}(\widehat{\lambda/\mu})$ and put $r_i = S(T_0^{-1}(i))$ for $i \in [1, n]$. Suppose $r_i - r_j = kn$ for some $i, j \in [1, n]$ and some $k \in \mathbb{Z}$. Then $S(T_0^{-1}(j) + k(m, -\ell)) = r_j + kn = r_i = S(T_0^{-1}(i))$, and hence $T_0^{-1}(j) + k(m, -\ell) = T_0^{-1}(i)$. This means $k = 0$ and $i = j$.

Let $r_i = \underline{r}_i + k_i n$ with $\underline{r}_i \in [1, n]$ and $k_i \in \mathbb{Z}$. Then we have shown that $\underline{r}_i \neq \underline{r}_j$ for $i, j \in [1, n]$ such that $i \neq j$. This ensures that there exists $x \in W$ such that $x(i) = \underline{r}_i$ for all $i \in [1, n]$. Putting $w := x \cdot \tau_{\epsilon_1}^{k_1} \tau_{\epsilon_2}^{k_2} \cdots \tau_{\epsilon_n}^{k_n}$, we have $w(i) = r_i$ for any $i \in [1, n]$ and hence $wT_0 = S$ on the fundamental domain λ/μ. This implies $wT_0 = S$.

It is easy to see that the choice of w for each S is unique, and hence the injectivity follows. ■

The following formula follows directly from the definition (3.17) of the action of $\dot{W}$.

Lemma 3.10. $T^{-1}(w^{-1}(i)) = (wT)^{-1}(i)$ for any $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, $w \in \dot{W}$, and $i \in \mathbb{Z}$. □

3.3 Content and weight

Let C denote the map from $\mathbb{Z}^2$ to $\mathbb{Z}$ given by $C(a, b) = b - a$ for $(a, b) \in \mathbb{Z}^2$.

For a tableau $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, define the map $C_T^{\widehat{\lambda/\mu}} : \mathbb{Z} \to \mathbb{Z}$ by

$$C_T^{\widehat{\lambda/\mu}}(i) = C\big(T^{-1}(i)\big) \quad (i \in \mathbb{Z}), \tag{3.20}$$

and call $C_T^{\widehat{\lambda/\mu}}$ the *content* of T. We simply denote $C_T^{\widehat{\lambda/\mu}}$ by C_T when (λ, μ) is fixed.

Lemma 3.11. Let $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$. Then

(i) $C_T(i + n) = C_T(i) - (\ell + m)$ for all $i \in \mathbb{Z}$;

(ii) $C_{wT}(i) = C_T(w^{-1}(i))$ for all $w \in \dot{W}$ and $i \in \mathbb{Z}$. □

Proof. (i) Put $(a, b) = T^{-1}(i) \in \widehat{\lambda/\mu}$. Then $T(a+m, b-\ell) = T(a, b) + n = i + n$. We have

$$\begin{aligned} C_T(i+n) &= C\big(T^{-1}(i+n)\big) = C(a+m, b-\ell) \\ &= (b-a) - (\ell+m) = C_T(i) - (\ell+m). \end{aligned} \tag{3.21}$$

(ii) The proof follows directly from Lemma 3.10. ■

For $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, we define $\zeta_T \in \dot{\mathfrak{h}}^*$ by

$$\zeta_T = \sum_{i=1}^{n} C_T(i)\epsilon_i + (\ell+m)c^*. \tag{3.22}$$

Then ζ_T belongs to the lattice

$$\dot{P} \overset{\text{def}}{=} P \oplus \mathbb{Z}c^* = \left(\bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i\right) \oplus \mathbb{Z}c^*. \tag{3.23}$$

Note that the action (2.14) of $\dot{W}$ on $\dot{\mathfrak{h}}^*$ preserves $\dot{P}$. Lemma 3.11 immediately implies the following.

Lemma 3.12. Let $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$. Then

(i) $\langle \zeta_T \mid \epsilon_i^\vee \rangle = C_T(i)$ for all $i \in \mathbb{Z}$;

(ii) $w(\zeta_T) = \zeta_{wT}$ for all $w \in \dot{W}$. □

3.4 The affine Weyl group and row increasing tableaux

Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$.

Definition 3.13. A tableau $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$ is said to be *row increasing* (resp., *column increasing*) if

$$\begin{aligned} &(a, b), (a, b+1) \in \widehat{\lambda/\mu} \Longrightarrow T(a, b) < T(a, b+1), \\ &\big(\text{resp., } (a, b), (a+1, b) \in \widehat{\lambda/\mu} \Longrightarrow T(a, b) < T(a+1, b)\big). \end{aligned} \tag{3.24}$$

A tableau $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$ which is row increasing and column increasing is called a *standard tableau* (or a *row-column increasing tableau*).

Denote by $\mathrm{Tab}^{\mathrm{R}}(\widehat{\lambda/\mu})$ (resp., $\mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$) the set of all row increasing (resp., standard) tableaux on $\widehat{\lambda/\mu}$.

For $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, put

$$I_{\lambda,\mu} = [1, n-1]/\{n_1, n_2, \ldots, n_{m-1}\}, \tag{3.25}$$

where $n_i = \sum_{j=1}^{i}(\lambda_j - \mu_j)$ for $i \in [1, m-1]$. We write $\dot{R}^+_{\lambda-\mu} = \dot{R}^+_{I_{\lambda,\mu}}$, $\dot{W}_{\lambda-\mu} = \dot{W}_{I_{\lambda,\mu}}$ and $\dot{W}^{\lambda-\mu} = \dot{W}^{I_{\lambda,\mu}}$. Note that $\dot{R}^+_{\lambda-\mu} \subseteq R^+$, and $\dot{W}_{\lambda-\mu} = W_{\lambda_1-\mu_1} \times W_{\lambda_2-\mu_2} \times \cdots \times W_{\lambda_m-\mu_m} \subseteq W$. Recall that the correspondence $\psi_T : \dot{W} \to \mathrm{Tab}(\widehat{\lambda/\mu})$ given by $w \mapsto wT$ is bijective (Proposition 3.9) for any $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$.

Proposition 3.14. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. Then

$$\psi_{T_0}^{-1}\left(\overset{R}{\mathrm{Tab}}(\widehat{\lambda/\mu})\right) = \dot{W}^{\lambda-\mu}, \tag{3.26}$$

or, equivalently, $\mathrm{Tab}^R(\widehat{\lambda/\mu}) = \dot{W}^{\lambda-\mu}T_0 = \{wT_0 \mid w \in \dot{W}^{\lambda-\mu}\}$. □

Proof. First we will prove $\dot{W}^{\lambda-\mu}T_0 \subseteq \mathrm{Tab}^R(\widehat{\lambda/\mu})$.

Take $(a, b), (a, b+1) \in \widehat{\lambda/\mu}$ and put $T_0(a, b) = i$. Then $T_0(a, b+1) = i+1$ and $\alpha_i \in \dot{R}^+_{\lambda-\mu}$. If $w \in \dot{W}^{\lambda-\mu}$, then we have $l(ws_i) > l(w)$. This means $w(\alpha_i) = \epsilon_{w(i)} - \epsilon_{w(i+1)} \in \dot{R}^+$. Hence, $w(i) < w(i+1)$, or, equivalently, $wT_0(a, b) < wT_0(a, b+1)$. Therefore, $wT_0 \in \mathrm{Tab}^R(\widehat{\lambda/\mu})$ for all $w \in \dot{W}^{\lambda-\mu}$.

Next, we will prove $\dot{W}^{\lambda-\mu}T_0 \supseteq \mathrm{Tab}^R(\widehat{\lambda/\mu})$.

For $T \in \mathrm{Tab}^R(\widehat{\lambda/\mu})$, take $w \in \dot{W}$ such that $wT_0 = T$. We have to show that $w \in \dot{W}^{\lambda-\mu}$. Let $\alpha_{ij} \in \dot{R}^+_{\lambda-\mu}$. Put $(a, b) = {T_0}^{-1}(i)$. Then ${T_0}^{-1}(j) = (a, b+j-i)$. Since wT_0 is row increasing, we have $wT_0(a, b) < wT_0(a, b+j-i)$ and hence $w(i) < w(j)$. This means that $w(\alpha_{ij}) \in \dot{R}^+$. Therefore, $\alpha_{ij} \notin R(w) = \dot{R}^+ \cap w^{-1}\dot{R}^-$. This proves $R(w) \cap \dot{R}^+_{\lambda-\mu} = \varnothing$ and hence $w \in \dot{W}^{\lambda-\mu}$. ■

3.5 The set of standard tableaux

The next lemma follows easily.

Lemma 3.15. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ and $T \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$. If $(a, b) \in \widehat{\lambda/\mu}$ and $(a+1, b+1) \in \widehat{\lambda/\mu}$, then $T(a+1, b+1) - T(a, b) > 1$. □

As a direct consequence of Lemma 3.15, we obtain the following result, which will be used in the next section.

Proposition 3.16. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ and $T, S \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$. If $C_T = C_S$, then $T = S$. □

Our next purpose is to describe the subset of $\dot{W}$ which corresponds to $\mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$ under the correspondence ψ_T ($T \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$).

Lemma 3.17. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. Let $w \in \dot{W}$ and $i \in [0, n-1]$ such that $wT_0 \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$ and $l(w) > l(s_i w)$. Then $s_i wT_0 \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$. □

Proof. We have $w \in \dot{W}^{\lambda-\mu}$ by Proposition 3.14. Put $x = s_i w$. Since $R(w) = R(x) \sqcup \{x^{-1}(\alpha_i)\}$, we have $x \in \dot{W}^{\lambda-\mu}$. Hence, $xT_0 \in \mathrm{Tab}^{R}(\widehat{\lambda/\mu})$.

Suppose $xT_0 \notin \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$. Then there exist (a,b) and $(a+1,b)$ in $\widehat{\lambda/\mu}$ such that

$$\begin{aligned} &xT_0(a,b) > xT_0(a+1,b),\\ &wT_0(a,b) < wT_0(a+1,b). \end{aligned} \tag{3.27}$$

This implies $xT_0(a,b) = i+1+kn$ and $xT_0(a+1,b) = i+kn$ for some $k \in \mathbb{Z}$.

On the other hand, we have $x^{-1}(\alpha_i) \in \dot{R}^{+}$ as $l(w) > l(x)$. Therefore, it follows that $x^{-1}(i) < x^{-1}(i+1)$ and hence $x^{-1}(i+kn) < x^{-1}(i+1+kn)$. Therefore, we have $T_0(a+1,b) < T_0(a,b)$, and this is a contradiction. ■

For $T \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$, put

$$\dot{Z}_T^{\widehat{\lambda/\mu}} = \left\{ w \in \dot{W} \mid \left\langle \zeta_T \mid \alpha^{\vee} \right\rangle \notin \{-1,1\}\, \forall \alpha \in R(w) \right\}. \tag{3.28}$$

Lemma 3.18. Let $(\lambda,\mu) \in \widehat{\mathcal{J}}^{n}_{m,\ell}$. Then

$$\dot{Z}_{T_0}^{\widehat{\lambda/\mu}} \subseteq \dot{W}^{\lambda-\mu}. \tag{3.29}$$

□

Proof. Take $w \in \dot{W}$ such that $w \notin \dot{W}^{\lambda-\mu}$. Then, it follows that there exists $j \in [0, n-1]$ such that $s_j \in \dot{W}_{\lambda-\mu}$ and $l(ws_j) < l(w)$. Then Lemma 2.2(ii) implies that $\alpha_j \in R(w)$. By $s_j \in \dot{W}_{\lambda-\mu}$, we have $\langle \zeta_{T_0} \mid \alpha_j^{\vee} \rangle = -1$. Hence, we have $w \notin \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$, and thus we proved that $\dot{Z}_{T_0}^{\widehat{\lambda/\mu}} \subseteq \dot{W}^{\lambda-\mu}$. ■

Theorem 3.19. Let $(\lambda,\mu) \in \widehat{\mathcal{J}}^{n}_{m,\ell}$ and $T \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$. Then

$$\psi_T^{-1}\left(\overset{RC}{\mathrm{Tab}}(\widehat{\lambda/\mu})\right) = \dot{Z}_T^{\widehat{\lambda/\mu}} \tag{3.30}$$

or, equivalently, $\mathrm{Tab}^{RC}(\widehat{\lambda/\mu}) = \dot{Z}_T^{\widehat{\lambda/\mu}} T$. □

Proof

Step 1. First we will prove the statement for the row reading tableau T_0, namely, we will prove $\mathrm{Tab}^{RC}(\widehat{\lambda/\mu}) = \dot{Z}_{T_0}^{\widehat{\lambda/\mu}} T_0$.

Let us see $\dot{Z}_{T_0}^{\widehat{\lambda/\mu}} T_0 \subseteq \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$, that is, $wT_0 \in \mathrm{Tab}^{RC}(\widehat{\lambda/\mu})$ for all $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$. We proceed by induction on $l(w)$.

If $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ with $l(w) = 0$, then $w = \pi^k$ for some $k \in \mathbb{Z}$ and it is obvious that wT_0 is row-column increasing. Suppose that wT_0 is row-column increasing for all $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ with $l(w) < k$.

Take $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ with $l(w) = k$. Note that $w \in \dot{W}^{\lambda-\mu}$ by Lemma 3.18. Take $x \in \dot{W}$ and $i \in [0, n-1]$ such that $w = s_i x$ and $l(w) = l(x) + 1$.

Note that $R(w) = R(x) \sqcup \{x^{-1}(\alpha_i)\}$, and hence $x \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$. By the induction hypothesis, we have $xT_0 \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. Suppose that $wT_0 \notin \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. Then wT_0 is not column increasing because wT_0 is row increasing by Proposition 3.14. Therefore, there exist $(a, b), (a+1, b) \in \widehat{\lambda/\mu}$ such that

$$\begin{aligned} &xT_0(a, b) < xT_0(a+1, b), \\ &wT_0(a, b) > wT_0(a+1, b). \end{aligned} \tag{3.31}$$

This implies that $xT_0(a, b) = i + kn$ and $xT_0(a+1, b) = i + 1 + kn$ for some $k \in \mathbb{Z}$. We have

$$\begin{aligned} \left\langle \zeta_{T_0} \mid x^{-1}\left(\alpha_i^\vee\right)\right\rangle &= \left\langle \zeta_{T_0} \mid x^{-1}\left(\alpha_{i+kn}^\vee\right)\right\rangle \\ &= C_{T_0}\left(x^{-1}(i+kn)\right) - C_{T_0}\left(x^{-1}(i+1+kn)\right) \\ &= b - a - \big(b - (a+1)\big) = 1. \end{aligned} \tag{3.32}$$

This contradicts that $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ and hence we have $wT_0 \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$.

Next, let us prove $\dot{Z}_{T_0}^{\widehat{\lambda/\mu}} T_0 \supseteq \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. We will show that $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ for all w such that $wT_0 \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$ by induction on $l(w)$. If $l(w) = 0$, then $R(w) = \varnothing$ and $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$. Let $k \in \mathbb{Z}_{\geq 1}$ and suppose that the statement is true for all w with $l(w) < k$.

Take $w \in \dot{W}$ such that $wT_0 \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$ and $l(w) = k$. Take $x \in \dot{W}$ and $i \in [0, n-1]$ such that $w = s_i x$ and $l(w) = l(x) + 1$. By Lemma 3.17, we have $xT_0 \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. By the induction hypothesis, we have $x \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$. Since $R(w) = R(x) \sqcup \{x^{-1}(\alpha_i^\vee)\}$, it is enough to prove

$$\sigma := \left\langle \zeta_{T_0} \mid x^{-1}\left(\alpha_i^\vee\right)\right\rangle = C_{T_0}\left(x^{-1}(i)\right) - C_{T_0}\left(x^{-1}(i+1)\right) \neq \pm 1. \tag{3.33}$$

We put $T = xT_0$ in the rest of the proof.

Suppose $\sigma = 1$. Put $(a, b) = T^{-1}(i)$. Then $T^{-1}(i+1) = (a+j+1, b+j)$ for some $j \in \mathbb{Z}$. If $j < 0$, then we have $(a, b-1) \in \widehat{\lambda/\mu}$ and $i+1 = T(a+j+1, b+j) \leq T(a, b-1) < T(a, b) = i$. This is a contradiction. If $j > 0$, then $(a+1, b) \in \widehat{\lambda/\mu}$ and $i + 1 > T(a+1, b) > i$. This is a contradiction too. Therefore, we must have $j = 0$ and hence $T^{-1}(i+1) = (a+1, b)$. But then we have

$$wT_0(a,b) = s_iT(a,b) = i+1 > i = s_iT(a+1,b) = wT_0(a+1,b) \tag{3.34}$$

and this contradicts the assumption $wT_0 \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. Therefore, $\sigma \neq 1$.

Suppose $\sigma = -1$. Put $(a,b) = T^{-1}(i)$. Then similar argument as above implies that $T^{-1}(i+1) = (a, b+1)$. This yields a contradiction too.

Therefore, we have $w \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ and thus we proved that $\mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu}) = \dot{Z}_{T_0}^{\widehat{\lambda/\mu}} T_0$.

Step 2. By Step 1, for each $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$, there exists $w_T \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ such that $T = w_T T_0$.

First we will show that $zw_T^{-1} \in \dot{Z}_T^{\widehat{\lambda/\mu}}$ for all $z \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$.

Assume that $zw_T^{-1} \notin \dot{Z}_T^{\widehat{\lambda/\mu}}$ for some $z \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$. Then there exists $\alpha \in R(zw_T^{-1})$ such that $\langle \zeta_T \mid \alpha^\vee \rangle = \pm 1$.

If $\alpha \in w_T R^+$, then putting $\beta = w_T^{-1}(\alpha)$, we have $\beta \in R(z)$ and $\langle \zeta_{T_0} \mid \beta^\vee \rangle = \langle \zeta_{w_T T_0} \mid w_T(\beta^\vee) \rangle = \pm 1$. This contradicts the choice $z \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$.

If $\alpha \notin w_T R^+$, then putting $\beta = -w_T^{-1}(\alpha)$, we have $\beta \in R(w_T)$ and $\langle \zeta_{T_0} \mid \beta^\vee \rangle = \pm 1$. This contradicts the choice $w_T \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$.

Therefore, $zw_T^{-1} \in \dot{Z}_T^{\widehat{\lambda/\mu}}$ for all $z \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$. Similarly, one can show that $zw_T \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ for all $z \in \dot{Z}_T^{\widehat{\lambda/\mu}}$. Hence, the correspondence $z \mapsto zw_T^{-1}$ gives a bijection from $\dot{Z}_{T_0}^{\widehat{\lambda/\mu}}$ to $\dot{Z}_T^{\widehat{\lambda/\mu}}$, whose inverse is given by $z \mapsto zw_T$. Therefore, we have

$$zT = zw_T T_0 \in \overset{\mathrm{RC}}{\mathrm{Tab}}(\widehat{\lambda/\mu}) \Longleftrightarrow zw_T \in \dot{Z}_{T_0}^{\widehat{\lambda/\mu}} \Longleftrightarrow z \in \dot{Z}_T^{\widehat{\lambda/\mu}}. \tag{3.35}$$

■

3.6 Content of standard tableaux

Let $n \in \mathbb{Z}_{\geq 2}$, $m \in \mathbb{Z}_{\geq 1}$, and $\ell \in \mathbb{Z}_{\geq 0}$. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}_{m,\ell}^n$ and $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. Put $\kappa = \ell + m$ and $F = C_T$. Then it is easy to check that the map $F : \mathbb{Z} \to \mathbb{Z}$ satisfies the following.

(C1) $F(i+n) = F(i) - \kappa$ for all $i \in \mathbb{Z}$.

(C2) For any $p \in \mathbb{Z}$ and $i, j \in F^{-1}(p)$ such that $i < j$ and $[i,j] \cap F^{-1}(p) = \{i,j\}$, there exist a unique $k_- \in F^{-1}(p-1)$ and a unique $k_+ \in F^{-1}(p+1)$ such that $i < k_- < j$, and $i < k_+ < j$, respectively.

Notice that condition (C1) implies that $\sharp F^{-1}(p)$ is finite for all $p \in \mathbb{Z}$.

Conversely, suppose that a map $F : \mathbb{Z} \to \mathbb{Z}$ satisfying conditions (C1) and (C2) is given. Then it can be seen that F is a content associated with some standard tableau on some periodic skew diagram, as in the following proposition.

Proposition 3.20. Let $n \in \mathbb{Z}_{\geq 2}$ and $\kappa \in \mathbb{Z}_{\geq 1}$. Suppose that the map $F : \mathbb{Z} \to \mathbb{Z}$ satisfies conditions (C1) and (C2) above. Then there exist $m \in [1, \kappa]$, $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\kappa-m}$, and $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$ such that $F = C_T$. □

Proof

Step 1. For $p \in F(\mathbb{Z}) \stackrel{\mathrm{def}}{=} \{F(i) \in \mathbb{Z} \mid i \in \mathbb{Z}\}$, put $d_p = \sharp F^{-1}(p)$, which (C1) implies is finite. Let $i_p^{(1)}, i_p^{(2)}, \ldots, i_p^{(d_p)}$ be the integers such that $i_p^{(1)} < i_p^{(2)} < \cdots < i_p^{(d_p)}$ and

$$F^{-1}(p) = \left\{ i_p^{(1)}, i_p^{(2)}, \ldots, i_p^{(d_p)} \right\}. \tag{3.36}$$

It follows from condition (C1) that $d_p = d_{p-\kappa}$ and

$$i_{p-\kappa}^{(1)} = i_p^{(1)} + n, \qquad i_{p-\kappa}^{(2)} = i_p^{(2)} + n, \ldots, i_{p-\kappa}^{(d_{p-\kappa})} = i_p^{(d_p)} + n \tag{3.37}$$

for all $p \in F(\mathbb{Z})$.

The following statement follows easily from condition (C2) and an induction argument (on j).

Claim 1. Let $p \in F(\mathbb{Z})$.

(i) If $p + 1 \in F(\mathbb{Z})$ and $i_p^{(1)} < i_{p+1}^{(1)}$, then $d_p - d_{p+1} = 0$ or 1, and it holds that

$$\begin{aligned} & i_p^{(j)} < i_{p+1}^{(j)} \quad \left(j \in [1, d_{p+1}]\right), \\ & i_p^{(j)} > i_{p+1}^{(j-1)} \quad \left(j \in [2, d_p]\right). \end{aligned} \tag{3.38}$$

(ii) If $p + 1 \in F(\mathbb{Z})$ and $i_p^{(1)} > i_{p+1}^{(1)}$, then $d_p - d_{p+1} = 0$ or -1, and it holds that

$$\begin{aligned} & i_p^{(j)} > i_{p+1}^{(j)} \quad \left(j \in [1, d_p]\right), \\ & i_p^{(j-1)} < i_{p+1}^{(j)} \quad \left(j \in [2, d_{p+1}]\right). \end{aligned} \tag{3.39}$$

(iii) If $p + 1 \notin F(\mathbb{Z})$, then $d_p = 1$.

Step 2. Fix $p_0 \in F(\mathbb{Z})$ and $r \in \mathbb{Z}$. We will define a subset $\Lambda = \Lambda_{p_0,r}$ of $\mathbb{Z}^2$ as follows.

For $p \in F(\mathbb{Z})$, define $\tilde{p}$ as the minimum number in $F(\mathbb{Z}) \cap \mathbb{Z}_{>p}$.

There exists a unique sequence $\{(a_p^{(1)}, b_p^{(1)})\}_{p \in F(\mathbb{Z})}$ in $\mathbb{Z}^2$ satisfying the initial condition

$$\left(a_{p_0}^{(1)}, b_{p_0}^{(1)}\right) = \left(r, p_0 + r\right) \tag{3.40}$$

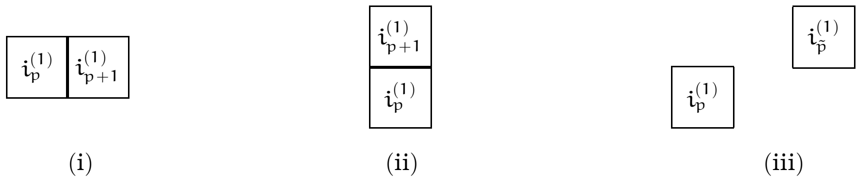

Figure 3.4 (i) $i_p^{(1)} < i_{p+1}^{(1)}$, (ii) $i_p^{(1)} > i_{p+1}^{(1)}$, and (iii) $\tilde p > p+1$.

and the recursion relation

$$\left(a_{\tilde p}^{(1)}, b_{\tilde p}^{(1)}\right) = \begin{cases} \left(a_p^{(1)}, b_p^{(1)}+1\right) & \text{if } \tilde p = p+1,\ i_p^{(1)} < i_{p+1}^{(1)},\ \text{Figure 3.4(i)}, \\ \left(a_p^{(1)}-1, b_p^{(1)}\right) & \text{if } \tilde p = p+1,\ i_p^{(1)} > i_{p+1}^{(1)},\ \text{Figure 3.4(ii)}, \\ \left(a_p^{(1)}-1, b_p^{(1)}+\tilde p - p - 1\right) & \text{if } \tilde p > p+1,\ \text{Figure 3.4(iii)}. \end{cases} \tag{3.41}$$

Put

$$\left(a_p^{(j)}, b_p^{(j)}\right) = \left(a_p^{(1)}+j-1, b_p^{(1)}+j-1\right) \quad \left(p \in F(\mathbb{Z}),\ j \in \left[2, d_p\right]\right), \tag{3.42}$$

and put

$$\Lambda = \left\{\left(a_p^{(j)}, b_p^{(j)}\right) \in \mathbb{Z}^2 \;\middle|\; p \in F(\mathbb{Z}),\ j \in \left[1, d_p\right]\right\}. \tag{3.43}$$

Note that $(a_p^{(1)}, b_p^{(1)})$ will be the most northwest box in $\widehat{\lambda/\mu}$ on the diagonal with content p.

Step 3. Now, we will check that the set Λ satisfies conditions (D1), (D2), and (D3) in Definition 3.2.

Check (D1). For $p \in F(\mathbb{Z})$, put

$$\ell_p = \sharp\left\{s \in [p, p+\kappa-1] \cap F(\mathbb{Z}) \mid \tilde s = s+1,\ i_s^{(1)} < i_{s+1}^{(1)}\right\}, \tag{3.44}$$

and put $m_p = \kappa - \ell_p$. Then $\ell_p, m_p \in [0, \kappa]$ and $m_p = a_p^{(1)} - a_{p+\kappa}^{(1)}$ by (3.41). Moreover, it follows from (3.37) that the number ℓ_p is independent of p, and so is $m = m_p$.

Since $b_p^{(1)} - a_p^{(1)} = p$, we have $(a_{p-\kappa}^{(1)}, b_{p-\kappa}^{(1)}) = (a_p^{(1)} + m, b_p^{(1)} - \kappa + m)$, and hence

$$\left(a_{p-\kappa}^{(j)}, b_{p-\kappa}^{(j)}\right) = \left(a_p^{(j)} + m, b_p^{(j)} - \kappa + m\right) \tag{3.45}$$

for all $j \in [1, d_p]$. Therefore, Λ satisfies condition (D1) with $\gamma := (m, -\kappa + m)$.

Check (D2). Put $E = \{(a_p^{(j)}, b_p^{(j)}) \mid p \in [1, \kappa] \cap F(\mathbb{Z}),\ j \in [1, d_p]\}$. Then E gives a fundamental domain of the action of $\mathbb{Z}\gamma$ on Λ, and the set E is in one to one correspondence with the set $F^{-1}([1, \kappa])$, by the definition of d_p. Hence, we have $\sharp E = \sharp F^{-1}([1, \kappa]) = n$ by (C1) and thus condition (D2) is checked.

Check (D3). Note that Claim 1 above implies that

$$(a, b), (a + 1, b + 1) \in \Lambda \Longrightarrow (a + 1, b), (a, b + 1) \in \Lambda. \tag{3.46}$$

Suppose that condition (D3) does not hold. Then there exist $(a, b) \in \Lambda$ and $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \setminus \{(0, 0), (1, 0), (0, 1)\}$ for which it holds that

$$(a + i', b + j') \in \Lambda \Longleftrightarrow (i', j') = (0, 0) \text{ or } (i, j). \tag{3.47}$$

Fix such (a, b) and (i, j).

First, suppose that $j - i = 0$. Then by (3.42), $i(= j)$ must be 1. This implies that $(a + 1, b), (a, b + 1) \in \Lambda$. This is a contradiction and hence $i - j \neq 0$.

Next, suppose that $j - i > 0$. Let $(a, b) = (a_s^{(r)}, b_s^{(r)})$ and $(a + i, b + j) = (a_p^{(k)}, b_p^{(k)})$. Note that $p - s = j - i > 0$.

If $k = 1$, then we have $a_p^{(1)} - a_s^{(r)} = i \geq 0$. On the other hand, it follows from the definitions (3.41), (3.42) of $\{(a_p^{(j)}, b_p^{(j)})\}_{p \in F(\mathbb{Z}), j \in [1, d_p]}$ that $a_s^{(r)} \geq a_s^{(1)} \geq a_p^{(1)}$ and the equalities hold only if $r = 1$ and $s, s + 1, \dots, p - 1 \in F(\mathbb{Z})$ and $i_s^{(1)} < i_{s+1}^{(1)} < \cdots < i_p^{(1)}$. This implies that $i = 0$ and $(a_{s+j'}^{(1)}, b_{s+j'}^{(1)}) = (a, b + j') \in \Lambda$ for all $j' \in [0, j]$. This is a contradiction.

If $k \neq 1$, then $(a_p^{(k-1)}, b_p^{(k-1)}) = (a + i - 1, b + j - 1) \in \Lambda$ and hence $(a + i, b + j - 1) \in \Lambda$ by (3.46). This is a contradiction since $(i, j - 1) \neq (0, 0), (i, j)$.

By similar argument, a contradiction is derived when $j - i < 0$. This means that (D3) holds for Λ, and hence $\Lambda = \mathcal{D}^n_{(m, -\kappa+m)}$. We show that Λ contains no empty rows. It is clear that Λ contains empty rows only if $m = 0$, that implies $F(\mathbb{Z}) = \mathbb{Z}$ and $i_p^{(1)} < i_{p+1}^{(1)}$ for all $p \in \mathbb{Z}$ by (3.41). But then it follows that $i_{p-\kappa}^{(1)} < i_p^{(1)}$ and this contradicts (3.37). Therefore, we have $\Lambda \in \mathcal{D}^{*n}_{(m, -\kappa+m)}$, or, equivalently, $\Lambda = \widehat{\lambda/\mu}$ for some $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m, \ell}$.

Step 4. Define the map $T : \Lambda \to \mathbb{Z}$ by $T(a_p^{(j)}, b_p^{(j)}) = i_p^{(j)}$. Obviously, we have $F = C \circ T^{-1}$. It follows from (3.37) and (3.45) that T is a tableau on Λ. Moreover, Claim 1 in Step 1 implies that T is row-column increasing, namely, $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. This completes the proof. ■

For $m \in \mathbb{Z}_{\geq 1}$, define an automorphism ω_m of $\mathbb{Z}^m$ by

$$\omega_m \cdot \lambda = \big(\lambda_m + \ell + 1, \lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_{m-1} + 1\big), \tag{3.48}$$

for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{Z}^m$. Let $\langle \omega_m \rangle$ denote the free group generated by ω_m, and let $\langle \omega_m \rangle$ act on $\mathbb{Z}^m \times \mathbb{Z}^m$ by $\omega_m \cdot (\lambda, \mu) = (\omega_m \cdot \lambda, \omega_m \cdot \mu)$ for $(\lambda, \mu) \in \mathbb{Z}^m \times \mathbb{Z}^m$. Note that $\langle \omega_m \rangle$ preserves the subsets $\widehat{\mathcal{J}}^n_{m,\ell}$ and $\widehat{\mathcal{J}}^{*n}_{m,\ell}$ of $\mathbb{Z}^m \times \mathbb{Z}^m$.

Proposition 3.21. Let $m, m' \in [1, n]$ and $\ell, \ell' \in \mathbb{Z}_{\geq 0}$. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\ell}$ and $(\eta, \nu) \in \widehat{\mathcal{J}}^{*n}_{m',\ell'}$. The following are equivalent:

(a) $C_T^{\widehat{\lambda/\mu}} = C_S^{\widehat{\eta/\nu}}$ for some $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$ and $S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\eta/\nu})$,

(b) $m = m'$, $\ell = \ell'$ and $\widehat{\lambda/\mu} = \widehat{\eta/\nu} + (r, r)$ for some $r \in \mathbb{Z}$,

(c) $m = m'$, $\ell = \ell'$ and $(\eta, \nu) = \omega_m^r \cdot (\lambda, \mu)$ for some $r \in \mathbb{Z}$. □

Proof. First we will prove (a)⇔(b).

It is easy to see that (b) implies (a). To see that (a) implies (b), recall the proof of Proposition 3.20, where the relations (3.41), (3.42) together with the initial condition (3.40) determine the periodic skew diagram $\Lambda_{p_0,r} = \{(a_p^{(j)}, b_p^{(j)}) \mid p \in F(\mathbb{Z}),\ j \in [1, d_p]\}$ and its period uniquely for each $p_0 \in F(\mathbb{Z})$ and $r \in \mathbb{Z}$.

Note that

$$\Lambda_{p_0,r'} = \Lambda_{p_0,r} + (r' - r, r' - r) \tag{3.49}$$

for $r, r' \in \mathbb{Z}$.

Put $F = C_T$. As in the proof of Proposition 3.20, we put $d_p = \sharp F^{-1}(p)$ for $p \in F(\mathbb{Z})$, and let $i_p^{(1)} < i_p^{(2)} < \cdots < i_p^{(d_p)}$ be the integers such that $F^{-1}(p) = \{i_p^{(1)}, i_p^{(2)}, \ldots, i_p^{(d_p)}\}$.

Put $(a_p^{(j)}, b_p^{(j)}) = T^{-1}(i_p^{(j)})$. Then it is easy to see that the sequence $\{(a_p^{(j)}, b_p^{(j)})\}_{p \in F(\mathbb{Z}), j \in [1, d_p]}$ satisfies the relations (3.41), (3.42). Therefore, we have

$$\widehat{\lambda/\mu} = \left\{T^{-1}\left(i_p^{(j)}\right) \mid p \in F(\mathbb{Z}),\ j \in \left[1, d_p\right]\right\} = \Lambda_{p_0,r} \tag{3.50}$$

for some $p_0 \in F(\mathbb{Z})$ and $r \in \mathbb{Z}$. Similarly, we have $\widehat{\eta/\nu} = \Lambda_{p_0,r'}$ for some $r' \in \mathbb{Z}$ (with the same $p_0 \in F(\mathbb{Z})$). Now, it follows from (3.49) that (a) implies (b).

The equivalence (b)⇔(c) follows from Proposition 3.4 and the formula

$$\widehat{\omega_m^r \cdot \lambda / \omega_m^r \cdot \mu} = \widehat{\lambda/\mu} - (r, r) \quad (r \in \mathbb{Z}), \tag{3.51}$$

which is verified by a simple calculation. ■

4 Representations of the double affine Hecke algebra

Let $\mathbb{F}$ denote a field whose characteristic is not equal to 2.

4.1 Double affine Hecke algebra of type A

Let $q \in \mathbb{F}$.

The double affine Hecke algebra was introduced by Cherednik [2, 4].

Definition 4.1. Let $n \in \mathbb{Z}_{\geq 2}$.

(i) The *double affine Hecke algebra* $\ddot{H}_n(q)$ of GL_n is the unital associative algebra over $\mathbb{F}$ defined by the following generators and relations:

(1) generators:

$$t_0, t_1, \ldots, t_{n-1}, \pi^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, \xi^{\pm 1}; \tag{4.1}$$

(2) relations for $n \geq 3$:

$$\begin{aligned}
&(t_i - q)(t_i + 1) = 0 \quad (i \in [0, n-1]),\\
&t_i t_j t_i = t_j t_i t_j \quad (j \equiv i \pm 1 \bmod n),\\
&t_i t_j = t_j t_i \quad (j \not\equiv i \pm 1 \bmod n),\\
&\pi\pi^{-1} = \pi^{-1}\pi = 1,\\
&\pi t_i \pi^{-1} = t_{i+1} \quad (i \in [0, n-2]), \qquad \pi t_{n-1}\pi^{-1} = t_0,\\
&x_i x_i^{-1} = x_i^{-1} x_i = 1 \quad (i \in [1, n]),\\
&x_i x_j = x_j x_i \quad (i, j \in [1, n]),\\
&t_i x_i t_i = q x_{i+1} \quad (i \in [1, n-1]), \qquad t_0 x_n t_0 = \xi^{-1} q x_1,\\
&t_i x_j = x_j t_i \quad (j \not\equiv i, i+1 \bmod n),\\
&\pi x_i \pi^{-1} = x_{i+1} \quad (i \in [1, n-1]), \qquad \pi x_n \pi^{-1} = \xi^{-1} x_1,\\
&\xi\xi^{-1} = \xi^{-1}\xi = 1, \qquad \xi^{\pm 1} h = h \xi^{\pm 1} \quad (h \in \ddot{H}_n(q));
\end{aligned} \tag{4.2}$$

(3) relations for $n = 2$:

$$\begin{aligned}
&(t_i - q)(t_i + 1) = 0 \quad (i \in [0, 1]),\\
&\pi\pi^{-1} = \pi^{-1}\pi = 1, \qquad \pi t_0 \pi^{-1} = t_1, \qquad \pi t_1 \pi^{-1} = t_0,\\
&x_i x_i^{-1} = x_i^{-1} x_i = 1 \quad (i \in [1, 2]), \qquad x_1 x_2 = x_2 x_1,\\
&t_1 x_1 t_1 = q x_2, \qquad t_0 x_2 t_0 = \xi^{-1} q x_1,\\
&\pi x_1 \pi^{-1} = x_2, \qquad \pi x_2 \pi^{-1} = \xi^{-1} x_1,\\
&\xi\xi^{-1} = \xi^{-1}\xi = 1, \qquad \xi^{\pm 1} h = h \xi^{\pm 1} \quad (h \in \ddot{H}_2(q)).
\end{aligned} \tag{4.3}$$

(ii) Define the *affine Hecke algebra* $\dot{H}_n(q)$ of GL_n as the subalgebra of $\ddot{H}_n(q)$ generated by $\{t_0, t_1, \dots, t_{n-1}, \pi^{\pm 1}\}$.

Remark 4.2. It is known that the subalgebra of $\ddot{H}_n(q)$ generated by

$$\{t_1, t_2, \dots, t_{n-1}, x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}\} \tag{4.4}$$

is also isomorphic to $\dot{H}_n(q)$.

For $\nu = \sum_{i=1}^{n} \nu_i \epsilon_i + \nu_c c^* \in \dot{P}$, put

$$x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n} \xi^{\nu_c}. \tag{4.5}$$

Let $\mathfrak{X}$ denote the commutative group $\{x^\nu \mid \nu \in \dot{P}\} \subseteq \ddot{H}_n(q)$. The group algebra $\mathbb{F}[\mathfrak{X}] = \mathbb{F}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, \xi^{\pm 1}]$ is a commutative subalgebra of $\ddot{H}_n(q)$.

For $w \in \dot{W}$ with a reduced expression $w = \pi^r s_{i_1} s_{i_2} \cdots s_{i_k}$, put

$$t_w = \pi^r t_{i_1} t_{i_2} \cdots t_{i_k}. \tag{4.6}$$

Then t_w does not depend on the choice of the reduced expression, and $\{t_w\}_{w \in \dot{W}}$ forms a basis of the affine Hecke algebra $\dot{H}_n(q) \subset \ddot{H}_n(q)$.

It is easy to see that $\{t_w x^\nu\}_{w \in \dot{W}, \nu \in \dot{P}}$ and $\{x^\nu t_w\}_{w \in \dot{W}, \nu \in \dot{P}}$, respectively, form basis of $\ddot{H}_n(q)$. In particular, we have following.

Proposition 4.3. $\ddot{H}_n(q) = \dot{H}_n(q)\mathbb{F}[\mathfrak{X}] = \mathbb{F}[\mathfrak{X}]\dot{H}_n(q)$. □

Define an element ϕ_i of $\ddot{H}_n(q)$ by

$$\phi_i = t_i\left(1 - x^{\alpha_i}\right) + 1 - q \quad \left(i \in [0, n-1]\right). \tag{4.7}$$

By direct calculations, we have the following.

Lemma 4.4. The following hold in $\ddot{H}_n(q)$:

$$\begin{aligned} &\phi_i \phi_j = \phi_j \phi_i \quad \left(i, j \in [0, n-1],\ j \not\equiv i \pm 1 \bmod n\right), \\ &\phi_i \phi_j \phi_i = \phi_j \phi_i \phi_j \quad \left(i, j \in [0, n-1],\ j \equiv i \pm 1 \bmod n\right), \\ &\phi_i^2 = \left(1 - q x^{\alpha_i}\right)\left(1 - q x^{-\alpha_i}\right) \quad \left(i \in [0, n-1]\right). \end{aligned} \tag{4.8}$$

□

For $w \in \dot{W}$ with a reduced expression $w = \pi^r s_{i_1} s_{i_2} \cdots s_{i_k}$, put

$$\phi_w = \pi^r \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k}. \tag{4.9}$$

Then ϕ_w does not depend on the choice of the reduced expression by Lemma 4.4. For an $\ddot{H}_n(q)$-module M, the element $\phi_w \in \ddot{H}_n(q)$ is regarded as a linear operator on M, and ϕ_w is called an *intertwining operator*.

The following formula follows easily.

Lemma 4.5. $\phi_w x^\nu = x^{w(\nu)}\phi_w$ for any $w \in \dot{W}$ and $\nu \in \dot{P}$. □

Lemma 4.6. For $w \in \dot{W}$,

$$\phi_w = t_w \prod_{\alpha \in R(w)} (1 - x^\alpha) + \sum_{y \in \dot{W}, y \prec w} t_y f_y \tag{4.10}$$

for some $f_y \in \mathbb{F}[\mathfrak{X}]$. □

Proof. The proof follows from the expression (2.17) of $R(w)$ and induction on $l(w)$. ■

Let $\mathfrak{X}^*$ denote the set of characters of $\mathfrak{X}$:

$$\mathfrak{X}^* = \mathrm{Hom}_{\mathrm{group}}(\mathfrak{X}, \mathrm{GL}_1(\mathbb{F})). \tag{4.11}$$

Consider the correspondence $\dot{P} \to \mathfrak{X}^*$ which maps $\zeta \in \dot{P}$ to the character $q^\zeta \in \mathfrak{X}^*$ defined by

$$q^\zeta(x_i) = q^{\langle \zeta | \epsilon_i^\vee \rangle} \quad (i \in [1, n]), \qquad q^\zeta(\xi) = q^{\langle \zeta | c \rangle}, \tag{4.12}$$

or, equivalently, defined by $q^\zeta(x^\nu) = q^{\langle \zeta | \nu^\vee \rangle}$ $(\nu \in \dot{P})$. Through this correspondence, $\dot{P}$ is identified with the subset

$$\{\chi \in \mathfrak{X}^* \mid \chi(x^\nu) \in q^{\mathbb{Z}} \ (\forall \nu \in \dot{P})\} \tag{4.13}$$

of $\mathfrak{X}^*$, where $q^{\mathbb{Z}} = \{q^r \mid r \in \mathbb{Z}\}$.

For an $\ddot{H}_n(q)$-module M and $\zeta \in \dot{P}$, define the weight space M_ζ and the generalized weight space M_ζ^{gen} of weight ζ with respect to the action of $\mathbb{F}[\mathfrak{X}]$ by

$$\begin{aligned} M_\zeta &= \left\{ v \in M \;\middle|\; (x^\nu - q^{\langle \zeta | \nu^\vee \rangle}) v = 0 \text{ for any } \nu \in \dot{P} \right\}, \\ M_\zeta^{\mathrm{gen}} &= \bigcup_{k \geq 1} \left\{ v \in M \;\middle|\; (x^\nu - q^{\langle \zeta | \nu^\vee \rangle})^k v = 0 \text{ for any } \nu \in \dot{P} \right\}. \end{aligned} \tag{4.14}$$

For an $\ddot{H}_n(q)$-module M, an element $\zeta \in \dot{P}$ is called a weight of M if $M_\zeta \neq 0$, and an element $v \in M_\zeta$ (resp., M_ζ^{gen}) is called a weight vector (resp., generalized weight vector) of weight ζ.

The following statement can be verified by direct calculations.

Proposition 4.7. Let M be an $\ddot{H}_n(q)$-module. Let $\zeta \in \dot{P}$ and $v \in M_\zeta$. Then the following hold, for all $w \in \dot{W}$:

(i) $\phi_w M_\zeta \subseteq M_{w(\zeta)}$ and $\phi_w M_\zeta^{\mathrm{gen}} \subseteq M_{w(\zeta)}^{\mathrm{gen}}$;

(ii) $\phi_{w^{-1}}\phi_w v = \prod_{\alpha\in R(w)} (1 - q^{1+\langle\zeta|\alpha^\vee\rangle})(1 - q^{1-\langle\zeta|\alpha^\vee\rangle})v$. □

For $\zeta \in \dot{P}$, put

$$\dot{\mathcal{Z}}_\zeta = \left\{w \in \dot{W} \mid \left\langle\zeta|\alpha^\vee\right\rangle \notin \{-1, 1\}\, \forall\alpha \in R(w)\right\}. \tag{4.15}$$

Note that $\dot{\mathcal{Z}}_{\zeta_T} = \dot{Z}_T^{\widehat{\lambda/\mu}}$ for $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ and $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$.

As a direct consequence of Proposition 4.7, we have the following.

Proposition 4.8. Suppose that q is not a root of 1. Let M be an $\ddot{H}_n(q)$-module and $\zeta \in \dot{P}$. For $w \in \dot{\mathcal{Z}}_\zeta$, the map

$$\phi_w : M_\zeta \longrightarrow M_{w(\zeta)} \tag{4.16}$$

is a linear isomorphism. □

4.2 $\mathfrak{X}$-semisimple modules

Remark 4.9. Throughout Section 4.2, the lemmas and propositions are still true and require almost no modification of their statements or proofs, even if κ is not an integer or if q is a root of unity. However, we impose these restrictions so that the combinatorics developed in Section 3 describes the structure of the $\mathfrak{X}$-semisimple modules. When we relax the condition $\kappa \in \mathbb{Z}$ but still require q generic, one can extend the combinatorial description with appropriate reformulation.

Fix $n \in \mathbb{Z}_{\geq 2}$. Let $q \in \mathbb{F}$ and suppose that q is not a root of 1.

Fix $\kappa \in \mathbb{Z}$ and put $P_\kappa = P + \kappa c^* = \{\zeta \in \dot{P} \mid \langle\zeta \mid c\rangle = \kappa\}$.

Definition 4.10. Define $\mathcal{O}_\kappa^{\mathrm{ss}}(\ddot{H}_n(q))$ as the set consisting of those $\ddot{H}_n(q)$-modules M which is finitely generated and admits a decomposition

$$M = \bigoplus_{\zeta\in P_\kappa} M_\zeta \tag{4.17}$$

with $\dim M_\zeta < \infty$ for all $\zeta \in P_\kappa$.

We say that a module $M \in \mathcal{O}^{ss}_{\kappa}(\ddot{H}_n(q))$ is $\mathfrak{X}$-*semisimple*.

In the following, we will see some general properties of $\ddot{H}_n(q)$-modules in $\mathcal{O}^{ss}_{\kappa}(\ddot{H}_n(q))$. The results and argument used in the proofs are essentially the same as those for the affine Hecke algebra (see, e.g., [9]).

Lemma 4.11. Let $M \in \mathcal{O}^{ss}_{\kappa}(\ddot{H}_n(q))$. Let $i \in [0, n-1]$ and let $\zeta \in P_\kappa$ be such that $\langle \zeta \mid \alpha_i^\vee \rangle = 0$. Then $M_\zeta = \{0\}$. □

Proof. Suppose that there exists $v \in M_\zeta \setminus \{0\}$. Then we have

$$\begin{aligned} &\left(x^{\alpha_i} - 1\right) t_i v = 2(1-q)v \neq 0, \\ &\left(x^{\alpha_i} - 1\right)^2 t_i v = 0. \end{aligned} \tag{4.18}$$

This implies $t_i v \in M_\zeta^{\mathrm{gen}} \setminus M_\zeta$, which contradicts the assumption $M = \bigoplus_{\zeta \in P_\kappa} M_\zeta$. ■

Lemma 4.12. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}^{ss}_{\kappa}(\ddot{H}_n(q))$. Let v be a nonzero weight vector of L. Then $L = \sum_{w \in \dot{W}} \mathbb{F} \phi_w v$. □

Proof. Put $N = \sum_{w \in \dot{W}} \mathbb{F} \phi_w v \subseteq L$. Since $L = \sum_{w \in \dot{W}} \mathbb{F} t_w v$ by Proposition 4.3, it is enough to prove that $t_w v \in N$ for all $w \in \dot{W}$. We proceed by induction on $l(w)$.

It is clear that $t_w v \in N$ for w of length zero.

Let $k \in \mathbb{Z}_{\geq 1}$ and suppose that $t_w v \in N$ for all $w \in \dot{W}$ with $l(w) < k$. Take $w \in \dot{W}$ with a reduced expression $w = \pi^r s_{i_1} s_{i_2} \cdots s_{i_k}$ (and hence $l(w) = k$). By Lemma 4.6, we have $\phi_w v = \sum_{x \in \dot{W}, x \preceq w} g_{wx} t_x v$ with some coefficients $g_{wx} \in \mathbb{F}$.

If $g_{ww} \neq 0$, then $t_w v = g_{ww}^{-1} (\phi_w v - \sum_{x \prec w} g_{wx} t_x v) \in N$.

Suppose $g_{ww} = 0$. By Lemma 4.6, this means

$$\prod_{\alpha \in R(w)} \left(1 - q^{\langle \zeta | \alpha^\vee \rangle}\right) = 0, \tag{4.19}$$

where $\zeta \in P_\kappa$ is the weight of v. Hence, there exists $p \in [1, k]$ such that

$$\prod_{\alpha \in R(y)} \left(1 - q^{\langle \zeta | \alpha^\vee \rangle}\right) \neq 0, \qquad \prod_{\alpha \in R(s_{i_p} y)} \left(1 - q^{\langle \zeta | \alpha^\vee \rangle}\right) = 0, \tag{4.20}$$

where $y = s_{i_{p+1}} s_{i_{p+2}} \cdots s_{i_k}$. This implies $\langle \zeta \mid y^{-1}(\alpha_{i_p}^\vee) \rangle = \langle y(\zeta) \mid \alpha_{i_p}^\vee \rangle = 0$. By Lemma 4.11, we have $L_{y(\zeta)} = 0$ and hence $\phi_y v = 0$.

Let $\phi_y v = \sum_{x\in\dot{W}, x\preceq y} g_{yx} t_x v$ with $g_{yx} \in \mathbb{F}$. Multiplying the equality $\phi_y v = 0$ by $\pi^r t_{i_1} t_{i_2} \cdots t_{i_p}$, we have

$$g_{yy} t_w v = -\sum_{x\prec y} g_{yx} \pi^r t_{i_1} t_{i_2} \cdots t_{i_p} t_x v. \tag{4.21}$$

Note that $g_{yy} = \prod_{\alpha\in R(y)}(1 - q^{\langle\zeta|\alpha^\vee\rangle}) \neq 0$ by (4.20), and it is easy to verify that the right-hand side of (4.21) is in N using the induction hypothesis. Therefore, $t_w v \in N$. ■

For $\zeta \in P_\kappa$, let $\dot{W}[\zeta]$ denote the stabilizer of ζ:

$$\dot{W}[\zeta] = \{w \in \dot{W} \mid w(\zeta) = \zeta\}. \tag{4.22}$$

Lemma 4.13. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$. Let ζ be a weight of L and let $v \in L_\zeta$. Then $\phi_w v = 0$ for all $w \in \dot{W}[\zeta] \setminus \{1\}$. □

Proof. Let $w \in \dot{W}[\zeta] \setminus \{1\}$ with a reduced expression $w = \pi^r s_{i_1} s_{i_2} \cdots s_{i_k}$.

Put $\dot{R}[\zeta] = \{\alpha \in \dot{R} \mid \langle\zeta \mid \alpha^\vee\rangle = 0\}$.

Then, $\dot{R}[\zeta]$ is a subroot system of $\dot{R}$ and $\dot{W}[\zeta]$ is the corresponding Coxeter group. Moreover, it follows that a system of positive (resp., negative) roots is given by $\dot{R}[\zeta] \cap \dot{R}^+$ (resp., $\dot{R}[\zeta] \cap \dot{R}^-$).

Therefore, for $w \in \dot{W}[\zeta] \setminus \{1\}$, there exists a reflection s_α ($\alpha \in \dot{R}[\zeta] \cap \dot{R}^+$) such that $w(\alpha) \in \dot{R}[\zeta] \cap \dot{R}^- \subseteq \dot{R}^-$.

Now, Lemma 2.2(ii) implies that there exists $p \in [0, n-1]$ such that $ws_\alpha = \pi^r s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_{p+1}} \cdots s_{i_k}$. Putting $y = s_{i_{p+1}} s_{i_{p+2}} \cdots s_{i_k}$, we have $\langle\zeta \mid y^{-1}(\alpha^\vee_{i_p})\rangle = \langle y(\zeta) \mid \alpha^\vee_{i_p}\rangle = 0$. Lemma 4.11 implies that $L_{y(\zeta)} = 0$ and hence $\phi_w v = \phi_{\pi^r s_{i_1} s_{i_2} \cdots s_{i_p}} \phi_y v = 0$. ■

Proposition 4.14. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$. Then $\dim L_\zeta \leq 1$ for all $\zeta \in P_\kappa$. □

Proof. The proof follows directly from Lemma 4.12 and Lemma 4.13. ■

Lemma 4.15. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$. Let ζ be a weight of L and let $i \in [0, n-1]$ such that $\langle\zeta \mid \alpha^\vee_i\rangle \in \{-1, 1\}$. Then $\phi_i v = 0$ for $v \in L_\zeta$. □

Proof. Suppose $\langle\zeta \mid \alpha^\vee_i\rangle = \pm 1$ and let $v \in L_\zeta \setminus \{0\}$. Suppose $\phi_i v \neq 0$. Put $\dot{W}' = \{w \in \dot{W} \mid ws_i \in \dot{W}[\zeta]\}$. Then it follows from Lemma 4.12 that

$$\sum_{w\in\dot{W}'} a_w \phi_w \phi_i v = v \tag{4.23}$$

for some $\{a_w \in \mathbb{F}\}_{w\in\dot{W}'}$.

For $w \in \dot{W}'$ such that $l(ws_i) < l(w)$, we have $\phi_w = \phi_{ws_i}\phi_i$. Proposition 4.7(ii) implies that

$$\phi_w\phi_i v = \phi_{s_i w}\phi_i^2 v = \phi_{s_i w}\left(1 - q^{1+\langle\zeta|\alpha_i^\vee\rangle}\right)\left(1 - q^{1-\langle\zeta|\alpha_i^\vee\rangle}\right)v \tag{4.24}$$

and it is 0 as $\langle\zeta \mid \alpha_i^\vee\rangle = \pm 1$.

For $w \in \dot{W}'$ such that $l(ws_i) > l(w)$, we have $\phi_w\phi_i v = \phi_{ws_i} v = 0$ by Lemma 4.13. Therefore, the left-hand side of (4.23) is 0 and this is a contradiction. ■

4.3 Representations associated with periodic skew diagrams

In the rest of this paper, we always assume that q is not a root of 1.

Let $n \in \mathbb{Z}_{\geq 2}$, $m \in \mathbb{Z}_{\geq 1}$, and $\ell \in \mathbb{Z}_{\geq 0}$.

For $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$, set

$$\ddot{V}(\lambda,\mu) = \bigoplus_{T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})} \mathbb{F}v_T. \tag{4.25}$$

Define linear operators $\tilde{x}_i$ $(i \in [1, n])$, $\tilde{\pi}$, and $\tilde{t}_i$ $(i \in [0, n-1])$ on $\ddot{V}(\lambda,\mu)$ by

$$\begin{aligned} &\tilde{x}_i v_T = q^{C_T(i)} v_T, \\ &\tilde{\pi} v_T = v_{\pi T}, \\ &\tilde{t}_i v_T = \begin{cases} \dfrac{1 - q^{1+\tau_i}}{1 - q^{\tau_i}} v_{s_i T} - \dfrac{1-q}{1-q^{\tau_i}} v_T & \text{if } s_i T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu}), \\ -\dfrac{1-q}{1-q^{\tau_i}} v_T & \text{if } s_i T \notin \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu}), \end{cases} \end{aligned} \tag{4.26}$$

where

$$\tau_i = C_T(i) - C_T(i+1) = \langle\zeta_T \mid \alpha_i^\vee\rangle \quad \big(i \in [0, n-1]\big). \tag{4.27}$$

The following lemma is easy and ensures that the operator $\tilde{t}_i$ is well defined.

Lemma 4.16. $C_T(i) - C_T(i+1) \neq 0$ for any $i \in [0, n-1]$ and $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. □

Theorem 4.17. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$. There exists an algebra homomorphism $\theta_{\lambda,\mu} : \ddot{H}_n(q) \to \mathrm{End}_{\mathbb{F}}(\ddot{V}(\lambda,\mu))$ such that

$$\begin{aligned} &\theta_{\lambda,\mu}(t_i) = \tilde{t}_i \quad \big(i \in [0, n-1]\big), \qquad \theta_{\lambda,\mu}(\pi) = \tilde{\pi}, \\ &\theta_{\lambda,\mu}(x_i) = \tilde{x}_i \quad \big(i \in [1, n]\big), \qquad \theta_{\lambda,\mu}(\xi) = q^{\ell+m}. \end{aligned} \tag{4.28}$$

□

Proof. The defining relations of $\ddot{H}_n(q)$ can be verified by direct calculations (see [9], for a sample of calculation for the affine Hecke algebra). ■

Note that the $\ddot{H}_n(q)$-module $\ddot{V}(\lambda, \mu)$ for $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$ belongs to $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$ with $\kappa = \ell + m$.

Theorem 4.18. Let $(\lambda, \mu) \in \widehat{\mathcal{J}}^n_{m,\ell}$.

(i) $\ddot{V}(\lambda, \mu) = \bigoplus_{T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})} \ddot{V}(\lambda, \mu)_{\zeta_T}$, and $\ddot{V}(\lambda, \mu)_{\zeta_T} = \mathbb{F} v_T$ for all $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$.

(ii) The $\ddot{H}_n(q)$-module $\ddot{V}(\lambda, \mu)$ is irreducible. □

Proof. (i) The proof follows directly from Proposition 3.16.

(ii) Let N be a nonzero submodule of $\ddot{V}(\lambda, \mu)$. Since N contains at least one weight vector, we can assume that $v_T \in N$ for some $T \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$.

Let $S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. By Theorem 3.19, there exists $w_S \in \dot{Z}_T^{\widehat{\lambda/\mu}}$ such that $S = w_S T$. Put $\tilde{v}_S = \phi_{w_S} v_T \in N$. Since the intertwining operator

$$\phi_{w_S} : \ddot{V}(\lambda, \mu)_{\zeta_T} \longrightarrow \ddot{V}(\lambda, \mu)_{w_S(\zeta_T)} = \ddot{V}(\lambda, \mu)_{\zeta_S} \tag{4.29}$$

is a linear isomorphism by Proposition 4.8, we have $\tilde{v}_S \in \ddot{V}(\lambda, \mu)_{\zeta_S} \setminus \{0\}$.

Now, it follows from (i) that $\bigoplus_{S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})} \mathbb{F}\tilde{v}_S = \ddot{V}(\lambda, \mu)$. Therefore, we have $N \supseteq \ddot{V}(\lambda, \mu)$ and hence $N = \ddot{V}(\lambda, \mu)$. Therefore, $\ddot{V}(\lambda, \mu)$ is irreducible. ■

4.4 Classification of $\mathfrak{X}$-semisimple modules

Fix $n \in \mathbb{Z}_{\geq 2}$ and $\kappa \in \mathbb{Z}_{\geq 1}$. Let $q \in \mathbb{F}$ and suppose that q is not a root of 1.

Our next and final purpose is to show that the modules $\ddot{V}(\lambda, \mu)$ we constructed in Section 4.3 exhausts all irreducible modules in $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$.

Lemma 4.19. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}^{ss}_\kappa(\ddot{H}_n(q))$. For any weight $\zeta \in P_\kappa$ of L and $i, j \in \mathbb{Z}$ such that $i < j$ and

$$\langle \zeta \mid \alpha^\vee_{ij} \rangle = 0, \tag{4.30}$$

there exist $k_+ \in [i+1, j-1]$ and $k_- \in [i+1, j-1]$ such that

$$\langle \zeta \mid \alpha^\vee_{ik_+} \rangle = -1, \qquad \langle \zeta \mid \alpha^\vee_{ik_-} \rangle = 1, \tag{4.31}$$

respectively. □

Proof. We proceed by induction on $j-i$.

For any weight ζ of L and $i \in \mathbb{Z}$, we have $\langle \zeta \mid \alpha_i^\vee \rangle \neq 0$ by Lemma 4.11. Therefore, we have nothing to prove when $j-i=1$. Let $r>1$ and assume that the statement holds when $j-i<r$.

In order to complete the induction step, it is enough to prove the existence of $k_\pm$ for a weight ζ of L and $i,j \in \mathbb{Z}$ such that $j-i=r$ and

$$\left\{k \in [i+1, j-1] \mid \left\langle \zeta \mid \alpha_{ik}^\vee \right\rangle = 0\right\} = \varnothing. \tag{4.32}$$

Fix a nonzero weight vector $v \in L_\zeta$.

Case 1. Suppose$\langle \zeta \mid \alpha_i^\vee \rangle = \pm 1$ and $\langle \zeta \mid \alpha_{j-1}^\vee \rangle = \pm 1$. Then the statement holds with $k_\pm = j-1$ and $k_\mp = i+1$.

Case 2. Suppose $\langle \zeta \mid \alpha_i^\vee \rangle = -1$ and $\langle \zeta \mid \alpha_{j-1}^\vee \rangle = 1$. Then we have $\langle \zeta \mid \alpha_{i+1j-1}^\vee \rangle = 0$. If $i+1 \neq j-1$, then there exist $k'_- \in [i+1, j-1]$ such that $\langle \zeta \mid \alpha_{i+1k'_-}^\vee \rangle = 1$, and hence $\langle \zeta \mid \alpha_{ik'_-}^\vee \rangle = \langle \zeta \mid \alpha_i^\vee \rangle + \langle \zeta \mid \alpha_{i+1k'_-}^\vee \rangle = 0$. In this contradicts the choice (4.32) of i,j. Therefore we have $j-i=2$. This case, we have $\langle \zeta \mid \alpha_i^\vee \rangle = -1$ and $\langle \zeta \mid \alpha_{i+1}^\vee \rangle = 1$. Hence, Lemma 4.15 implies that $\phi_i v = 0$ and $\phi_{i+1} v = 0$, which gives $t_i v = qv$ and $t_{i+1} v = -v$, respectively. But then we have

$$-q^2 v = t_i t_{i+1} t_i v = t_{i+1} t_i t_{i+1} v = qv, \tag{4.33}$$

and this is a contradiction as q is not a root of 1. Therefore, this case is not possible.

Case 3. Suppose $\langle \zeta \mid \alpha_i^\vee \rangle = 1$ and $\langle \zeta \mid \alpha_{j-1}^\vee \rangle = -1$. A similar argument as in Case 2 implies that this case is not possible.

Case 4. Suppose $\langle \zeta \mid \alpha_i^\vee \rangle \neq \pm 1$. Then $\phi_i v \neq 0$ by Proposition 4.7 and hence $s_i(\zeta)$ is a weight of L. By $\langle s_i(\zeta) \mid \alpha_{i+1j}^\vee \rangle = 0$, the induction hypothesis implies that there exists $k_\pm \in [i+2, j-1]$ such that $\langle \zeta \mid \alpha_{ik_\pm}^\vee \rangle = \langle s_i(\zeta) \mid \alpha_{i+1k_\pm}^\vee \rangle = \mp 1$. Hence the statement holds.

Case 5. Suppose $\langle \zeta \mid \alpha_{j-1}^\vee \rangle \neq \pm 1$. Then $\phi_{j-1} v \neq 0$ and a similar argument as in Case 4 implies that there exists $k_\pm \in [i+1, j-2]$ such that $\langle \zeta \mid \alpha_{ik_\pm}^\vee \rangle = \mp 1$.

This completes the proof. ■

Theorem 4.20. Let $n \in \mathbb{Z}_{\geq 2}$ and $\kappa \in \mathbb{Z}_{\geq 1}$. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}_\kappa^{ss}(\ddot{H}_n(q))$. Then there exist $m \in [1, \kappa]$ and $(\lambda, \mu) \in \widehat{\mathcal{J}}^{*n}_{m,\kappa-m}$ such that $L \cong \ddot{V}(\lambda, \mu)$. □

Proof

Step 1. Let $\zeta \in P_\kappa$ be a weight of L. Define $F_\zeta : \mathbb{Z} \to \mathbb{Z}$ by $F_\zeta(i) = \langle \zeta \mid \epsilon_i^\vee \rangle$ $(i \in \mathbb{Z})$.

It is easy to see that F_ζ satisfies condition (C1) in Proposition 3.20, and the existence of $k_\pm$ in condition (C2) follows from Lemma 4.19. Note that the uniqueness of $k_\pm$ in (C2) follows automatically from the condition that $[i,j] \cap F_\zeta^{-1}(p) = \{i,j\}$, setting $p = F_\zeta(i)$ here. Suppose, without loss of generality, that there were another choice of $k'_\pm$, with $k_\pm < k'_\pm$. It follows $\langle \zeta \mid \alpha^\vee_{k_\pm k'_\pm} \rangle = 0$, and applying Lemma 4.19 here gives the existence of an i' between $k_\pm$ and $k'_\pm$ and hence $i < i' < j$ with $\langle \zeta \mid \alpha^\vee_{ii'} \rangle = 0$. This gives $i' \in F_\zeta^{-1}(p)$, a contradiction.

Therefore, Proposition 3.20 implies that there exist $m \in [1,n]$, $T \in \mathrm{Tab}(\widehat{\lambda/\mu})$, and $(\lambda,\mu) \in \widehat{\mathcal{J}}^{*n}_{m,\kappa-m}$ such that $F_\zeta = C_T$, or, equivalently, $\zeta = \zeta_T$.

Step 2. Recall that $\dot{\mathcal{Z}}_\zeta = \dot{Z}_T^{\widehat{\lambda/\mu}}$. Take $u \in L_\zeta \setminus \{0\}$. For each $w \in \dot{Z}_T^{\widehat{\lambda/\mu}}$, put

$$
\begin{aligned}
\sigma_w &= \prod_{\alpha \in R(w)} \left(1 - q^{1+\langle \zeta \mid \alpha^\vee \rangle}\right), \\
u_w &= \sigma_w^{-1} \phi_w u.
\end{aligned}
\tag{4.34}
$$

Here, note that $\sigma_w \neq 0$ and $u_w \neq 0$ for all $w \in \dot{Z}_T^{\widehat{\lambda/\mu}}$ by Proposition 4.8.

Put $N = \sum_{w \in \dot{\mathcal{Z}}_\zeta} \mathbb{F}\phi_w u = \sum_{w \in \dot{Z}_T^{\widehat{\lambda/\mu}}} \mathbb{F}u_w \subseteq L$. Since $u_w \in L_{\zeta_{wT}}$ and each weight space is linearly independent by Proposition 3.16, we have $N = \bigoplus_{w \in \dot{Z}_T^{\widehat{\lambda/\mu}}} \mathbb{F}u_w$.

By Theorem 3.19, one can define $w_S \in \dot{Z}_T^{\widehat{\lambda/\mu}}$ by $S = w_S T$ for all $S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$, and define a linear map $\rho : \ddot{V}(\lambda,\mu) \to L$ by $\rho(v_S) = u_{w_S}$ $(S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu}))$. It is obvious that ρ is injective and its image is N.

Let us see that ρ is an $\ddot{H}_n(q)$-homomorphism.

Let $w \in \dot{Z}_T^{\widehat{\lambda/\mu}}$. Let $i \in [0, n-1]$ be such that $l(s_i w) < l(w)$. Then we have $s_i w \in \dot{Z}_T^{\widehat{\lambda/\mu}}$ and $\sigma_w = (1 - q^{1+\langle \zeta \mid (s_i w)^{-1}(\alpha_i^\vee) \rangle})\sigma_{s_i w}$. Therefore,

$$
\begin{aligned}
\phi_i u_w &= \sigma_w^{-1} \phi_i \phi_w u = \sigma_w^{-1} \phi_i^2 \phi_{s_i w} u \\
&= \left(1 - q^{1+\langle s_i w(\zeta) \mid \alpha_i^\vee \rangle}\right)\left(1 - q^{1-\langle s_i w(\zeta) \mid \alpha_i^\vee \rangle}\right)\sigma_w^{-1} \phi_{s_i w} u \\
&= \left(1 - q^{1+\langle w(\zeta) \mid \alpha_i^\vee \rangle}\right) u_{s_i w}.
\end{aligned}
\tag{4.35}
$$

Let $i \in [0, n-1]$ be such that $l(s_i w) > l(w)$. If $s_i w \notin \dot{Z}_T^{\widehat{\lambda/\mu}}$, then $\langle \zeta \mid w^{-1}(\alpha_i^\vee) \rangle = \pm 1$ and hence $\phi_i u_w = 0$ by Lemma 4.15. If $s_i w \in \dot{Z}_T^{\widehat{\lambda/\mu}}$, then we have $\sigma_{s_i w} = (1 - q^{1-\langle \zeta \mid w^{-1}(\alpha_i^\vee) \rangle})\sigma_w$

and

$$\begin{aligned}\phi_i u_w &= \sigma_w^{-1}\phi_i\phi_w u = \sigma_w^{-1}\phi_{s_i w}u \\ &= \big(1 - q^{1+\langle w(\zeta)|\alpha_i^\vee\rangle}\big)u_{s_i w}.\end{aligned} \tag{4.36}$$

Therefore, in both cases, we have

$$\phi_i u_w = \begin{cases}\big(1 - q^{1+\langle w(\zeta)|\alpha_i^\vee\rangle}\big)u_{s_i w} & \Big(s_i w \in \dot{Z}_T^{\widehat{\lambda/\mu}}\Big), \\ 0 & \Big(s_i w \notin \dot{Z}_T^{\widehat{\lambda/\mu}}\Big).\end{cases} \tag{4.37}$$

This implies

$$\rho\big(t_i v_S\big) = t_i\rho\big(v_S\big) \quad \Big(i \in [0, n-1],\ S \in \overset{\mathrm{RC}}{\mathrm{Tab}}(\widehat{\lambda/\mu})\Big). \tag{4.38}$$

Moreover, it is easy to see that

$$\rho\big(x_i v_S\big) = x_i\rho(v_S) \quad (i \in [1,n]), \qquad \rho(\xi v_S) = \xi\rho(v_S), \qquad \rho(\pi v_S) = \pi\rho(v_S) \tag{4.39}$$

for all $S \in \mathrm{Tab}^{\mathrm{RC}}(\widehat{\lambda/\mu})$. Therefore, ρ is an $\ddot{H}_n(q)$-homomorphism and it gives an isomorphism $\ddot{V}(\lambda,\mu) \cong N$ of $\ddot{H}_n(q)$-modules. Since L is irreducible, we have $L = N \cong \ddot{V}(\lambda,\mu)$. ■

Corollary 4.21. Let L be an irreducible $\ddot{H}_n(q)$-module which belongs to $\mathcal{O}_\kappa^{ss}(\ddot{H}_n(q))$. Let $v \in L$ be a nonzero weight vector of weight $\zeta \in \dot{P}_\kappa$. Then

$$L = \bigoplus_{w \in \dot{\mathcal{Z}}_\zeta} \mathbb{F}\phi_w v, \tag{4.40}$$

and $\phi_w v \neq 0$ for all $w \in \dot{\mathcal{Z}}_\zeta$. □

Theorem 4.22. Let $m, m' \in \mathbb{Z}_{\geq 1}$ and $\ell, \ell' \in \mathbb{Z}_{\geq 0}$. Let $(\lambda,\mu) \in \widehat{\mathcal{J}}^{*n}_{m,\ell}$ and $(\eta,\nu) \in \widehat{\mathcal{J}}^{*n}_{m',\ell'}$. Then the following are equivalent:

(a) $\ddot{V}(\lambda,\mu) \cong \ddot{V}(\eta,\nu)$,

(b) $m = m'$, $\ell = \ell'$ and $\widehat{\lambda/\mu} = \widehat{\eta/\nu} + (r,r)$ for some $r \in \mathbb{Z}$,

(c) $m = m'$, $\ell = \ell'$ and $(\eta,\nu) = \omega_m^r \cdot (\lambda,\mu)$ for some $r \in \mathbb{Z}$. □

Proof. The proof follows from Step 1 in the proof of Theorem 4.20 and Proposition 3.21. ■

Let $\mathrm{Irr}\mathcal{O}_\kappa^{ss}(\ddot{H}_n(q))$ denote the set of isomorphism classes of all simple modules in $\mathcal{O}_\kappa^{ss}(\ddot{H}_n(q))$. Combining Theorems 4.20 and 4.22, we obtain the following classification theorem, which is announced in [5] in more general situation.

Corollary 4.23 (cf.[5])**.** Let $n \in \mathbb{Z}_{\geq 2}$ and $\kappa \in \mathbb{Z}_{\geq 1}$. The correspondences $(\lambda, \mu) \mapsto \widehat{\lambda/\mu}$ and $(\lambda, \mu) \mapsto \ddot{V}(\lambda, \mu)$ induce the following bijections, respectively:

$$\bigsqcup_{m \in [1,\kappa]} \mathcal{D}^{*n}_{(m,-\kappa+m)}/\mathbb{Z}(1,1) \xleftarrow{\sim} \bigsqcup_{m \in [1,\kappa]} \left(\widehat{\mathcal{J}}^{*n}_{m,\kappa-m}/\langle \omega_m \rangle\right) \xrightarrow{\sim} \mathrm{Irr}\mathcal{O}^{ss}_{\kappa}\left(\ddot{H}_n(q)\right). \tag{4.41}$$

□

Remark 4.24. We gave a direct and combinatorial proof for Theorems 4.20 and 4.22 and Corollary 4.23 based on the tableaux theory on periodic skew diagrams.

An alternative approach to prove these results is to use the result in [10, 11], where the classification of irreducible modules over $\ddot{H}_n(q)$ of a more general class is obtained. Actually, it is easy to see that the $\ddot{H}_n(q)$-module $\ddot{V}(\lambda, \mu)$ coincides with the unique simple quotient $\ddot{L}(\lambda, \mu)$ of the induced module $\ddot{M}(\lambda, \mu)$ with the notation in [10].

Remark 4.25. It is easy to derive the corresponding results for the degenerate affine Hecke algebra by a parallel argument.

Remark 4.26. There exists an algebra involution $\iota : \ddot{H}_n(q) \to \ddot{H}_n(q)$ such that

$$\begin{aligned} &\iota(t_i) = q t_i^{-1} \quad (i \in [0, n-1]), \qquad \iota(\pi) = \pi, \\ &\iota(x_i) = x_i^{-1} \quad (i \in [1, n]), \qquad \iota(\xi) = \xi^{-1}. \end{aligned} \tag{4.42}$$

The composition $\theta_{\lambda,\mu} \circ \iota : \ddot{H}_n(q) \to \ddot{V}(\lambda, \mu)$ gives an $\ddot{H}_n(q)$-module structure on $\ddot{V}(\lambda, \mu)$ on which ξ acts as a scalar $q^{-\ell-m}$. We let $\ddot{V}^{\iota}(\lambda, \mu)$ denote this $\ddot{H}_n(q)$-module. The correspondence $(\lambda, \mu) \mapsto \ddot{V}^{\iota}(\lambda, \mu)$ induces a bijection

$$\bigsqcup_{m \in [1,\kappa]} \left(\widehat{\mathcal{J}}^{*n}_{m,\kappa-m}/\langle \omega_m \rangle\right) \longrightarrow \mathrm{Irr}\mathcal{O}^{ss}_{-\kappa}\left(\ddot{H}_n(q)\right) \tag{4.43}$$

for all $\kappa \in \mathbb{Z}_{\geq 1}$.

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