Gram-Schmidt orthogonalization

`http://planetmath.org/encyclopedia/GramSchmidtOrthogonalization.html`.

Gram-Schmidt orthogonalization

Any set of linearly independent vectors $v_1, \dots, v_n$ can be converted into a set of orthogonal vectors $q_1, \dots, q_n$ by the Gram-Schmidt process. In three dimensions, v_1 determines a line; the vectors v_1 and v_2 determine a plane. The vector q_1 is the unit vector in the direction v_1. The (unit) vector q_2 lies in the plane of v_1, v_2, and is normal to v_1 (on the same side as v_2. The (unit) vector q_3 is normal to the plane of v_1, v_2, on the same side as v_3, etc.

In general, first set $u_1 = v_1$, and then each u_i is made orthogonal to the preceding $u_1, \dots u_{i-1}$ by subtraction of the projections of v_i in the directions of $u_1, \dots, u_{i-1}$:

$u_i = v_i - \sum_{j=1}^{i-1} \frac{u_j^T v_i}{u_j^T uj} u_j$

The i vectors u_i span the same subspace as the v_i. The vectors $q_i = u_i/||u_i||$ are orthonormal.

From `http://en.wikipedia.org/wiki/Gram-Schmidt_process`

GramSchmidt process

In mathematics and numerical analysis, the GramSchmidt process is a method for orthogonalizing a set of vectors in an inner product space. The GramSchmidt process takes a finite, linearly independent set $S = \{v_1, \dots, v_n\}$ and generates an orthogonal set $S' = \{u_1, \dots, u_n\}$ that spans the same subspace as S.

The GramSchmidt process

We define the projection operator by

$$\mathrm{proj}_{\mathbf{u}}\, \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}.$$

It projects the vector $\mathbf{v}$ orthogonally onto the vector $\mathbf{u}$.

The GramSchmidt process then works as follows:

$$
\begin{aligned}
\mathbf{u}_1 =& \mathbf{v}_1, \\
& \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 =& \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}\,\mathbf{v}_2, \\
& \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 =& \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}\,\mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_2}\,\mathbf{v}_3, \\
& \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 =& \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}\,\mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_2}\,\mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_3}\,\mathbf{v}_4, \\
& \mathbf{e}_4 = \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
& \vdots \vdots \\
\mathbf{u}_k =& \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}\,\mathbf{v}_k, \\
& \mathbf{e}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}
\end{aligned}
$$

The sequence $\{u_1, \ldots, u_k\}$ is the required system of orthogonal vectors, and the normalized vectors $\{e_1, \ldots, e_k\}$ form an orthonormal set.

To check that these formulas yield an orthogonal sequence, first compute $\langle u1, u2 \rangle$ by substituting the above formula for u2: you will get zero. Then use this to compute $\langle u1, u3 \rangle$ again by substituting the formula for u3: you will get zero. The general proof proceeds by mathematical induction.

Geometrically, this method proceeds as follows: to compute ui, it projects vi orthogonally onto the subspace U generated by $\{u_1, \ldots, u_{i-1}\}$ which is the same as the subspace generated by $\{v_1, \ldots, v_{i-1}\}$. The vector ui is then defined to be the difference between vi and this projection, guaranteed to be orthogonal to all of the vectors in the subspace U. The GramSchmidt process also applies to a linearly independent infinite sequence vii.

If GramSchmidt process applies to linearly dependent sequence, the output would give 0 vector on the i-th step, when vi linearly depends on previous $\{v_1, \ldots, v_{i-1}\}$.

Example

Consider the following set of vectors in $\mathbf{R}^2$ (with the conventional inner product)

$$S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}.$$

Now, perform GramSchmidt, to obtain an orthogonal set of vectors:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \operatorname{proj}_{\mathbf{u}_1} \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \operatorname{proj}_{\left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right)} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}.$$

We check that the vectors u1 and u2 are indeed orthogonal:
$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \left\langle \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix} \right\rangle = -\frac{6}{5} + \frac{6}{5} = 0.$

We can then normalize the vectors by dividing out their sizes as shown above:
$\mathbf{e}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \mathbf{e}_2 = \frac{1}{\sqrt{\frac{40}{25}}} \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$