Handout M150B
W09
Bilinear Forms

Let $\langle , \rangle$ be a bilinear form on the vector space V.

With respect to (wrt) the standard (std) basis $\langle , \rangle$ is associated to $A = [a_{ij}]$ if

$$\langle v, w \rangle = v^T A w \qquad \forall \; v, w \in V = \mathbb{F}^n$$

iff $\quad a_{ij} = \langle \underline{e_i}, \underline{e_j} \rangle$.

Recall $\underline{e_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith position.

Wrt basis $\mathcal{B}' = \{\underline{b_1}, \ldots, \underline{b_n}\}$, A' ~~is~~ **represents** $\langle , \rangle$ if

$$a_{ij} = \langle \underline{b_i}, \underline{b_j} \rangle .$$

See the COB handout: We write

$$[v]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \text{ if } \quad v = \sum_{i=1}^{n} \alpha_i \underline{b_i} .$$

$$P = P_{\mathcal{B}}^{\mathcal{B}'} \quad \text{if} \quad P[v]_{\mathcal{B}} = [v]_{\mathcal{B}'} .$$

Note $P_{\mathcal{B}}^{std}$ is the "easy" one

$$P_{std}^{\mathcal{B}} = \left(P_{\mathcal{B}}^{std}\right)^{-1}$$

$$P_{\mathcal{B}}^{std} = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix}$$

So $\quad \langle v, w \rangle = [v]_{\mathcal{B}'}^T A' [w]_{\mathcal{B}'}$.

But $[v]_{B'}^T A' [v]_{B'}$

$$= (P[v]_B)^T A' (P[v]_B)$$

$$= [v]_B^T P^T A' P [v]_B$$

So A, A' both represent $\langle , \rangle$ iff $\exists$ invertible P with $A = P^T A' P$

A is symmetric iff $\langle , \rangle$ is symmetric
($a_{ij} = a_{ji}$) ($\langle \underline{v}, \underline{w} \rangle = \langle \underline{w}, \underline{v} \rangle$)

Over $\mathbb{R}$/
$\langle , \rangle$ is symmetric & positive definite

iff

$\exists$ an ONB (wrt $\langle , \rangle$) for V
(via Gram Schmidt)

iff

I represents $\langle , \rangle$

iff

for any matrix A representing $\langle , \rangle$,
$\exists$ invertible P with $A = P^T P$