

Worksheet 1
Applying Sylow's Theorems Jan 5, 2009

Goal- Classify all groups G of order 15. Let s_3 = the number of Sylow-3-subgroups of order 3.

Let s_5 = the number of Sylow-5-subgroups of order 5.

1. What are all possible s_3 ?
2. What are all possible s_5 ?

Recall, Sylow III says $s_p \equiv 1 \pmod{p}$, $s_p \mid m$.

Recall, Sylow II says all Sylow p 's are conjugate.

3. For each possibility/case from part (a), how many elements of order 3 are there in G ?
4. For each possibility/case from part (b), how many elements of order 5 are there in G ?
5. Let H = any Sylow-3 subgroup of G . Is $H \triangleleft G$? (Why or why not? It will depend on each possibility/case from part (a).)
6. Let K = any Sylow-5 subgroup of G . Is $K \triangleleft G$?
7. What is $|H|$ =? Note H is cyclic. Write $H = \langle x \rangle$
What is $|K|$ =? Note K is cyclic. Write $K = \langle y \rangle$
8. What is $|H \cap K|$ =?
9. HK is a subgroup of G . WHY?
10. What is $|HK|$ =?
Recall $|HK| = \frac{|H||K|}{|H \cap K|}$.
11. We can write any $g \in G$ as $g = x^a y^b$. WHY?
12. Apply Proposition 2.8.6.
What can you say about $H \times K$? About G ?
13. Identify G up to isomorphism.

Goal- Classify all groups G of order 21. Do the same thing...

HINT: If $xyx^{-1} = x^a$ for some $a \in \mathbf{Z}$, note

$$(xyx^{-1})^2 = xy^2x^{-1} \quad \text{and more generally } (xyx^{-1})^n = xy^nx^{-1}$$

but $y^2xy^{-2} = y(xyx^{-1})y^{-1} = yx^ay^{-1} = (xyx^{-1})^a = (x^a)^a = x^{a^2}$
and so $y^nxy^{-n} = x^{a^n}$.