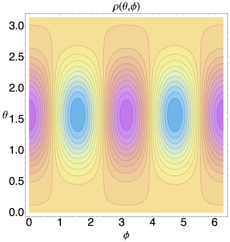
Gravitational turbulent instability of AdS

Jorge E. Santos



Santa Barbara - University of California

In collaboration with
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  - A poor's man approach: break down of perturbation theory - onset of interesting dynamics.

Anti-de Sitter spacetime - 1/2

Anti-de Sitter space is a maximally symmetric solution to

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}\left[R + \frac{(d-1)(d-2)}{L^2}\right],$$

which in global coordinates can be expressed as

$$\mathrm{d}s^2 \equiv \bar{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\left(\frac{r^2}{L^2}+1\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\frac{r^2}{L^2}+1} + r^2\mathrm{d}\Omega^2_{d-2}.$$

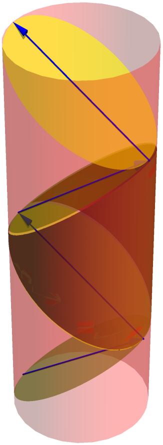
The Poincaré coordinates

$$\mathrm{d}s^2 = R^2(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}) + \frac{L^2\mathrm{d}R^2}{R^2}$$

do not cover the entire spacetime.

Anti-de Sitter spacetime - 2/2

Conformally, AdS looks like the interior of a cylinder



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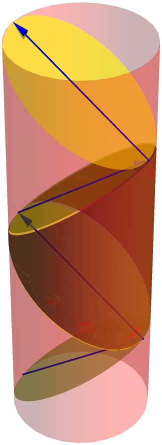

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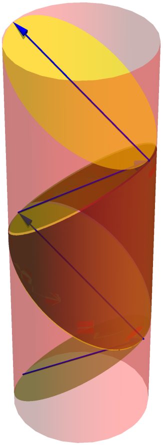

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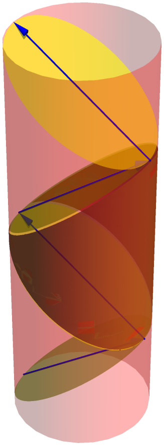

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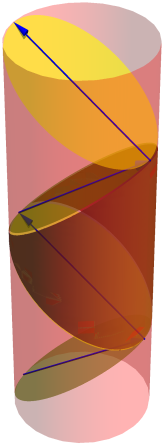

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- The instability described in this talk will occur in global AdS only.
- The dual field theory lives on $\mathbb{R}_t \times S^{d-2}$.
- With energy preserving boundary conditions, waves bounce off infinity and return in finite time.

Minkowski, dS and AdS spacetimes

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- The energy cascades from low to high frequency modes in a manner reminiscent of the onset of turbulence.

Folklore - 1/2

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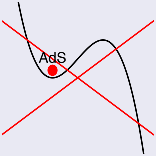

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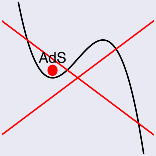

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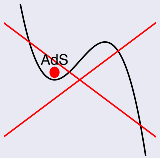
  - That is usually ruled out by arguing that waves disperse. This does not happen in AdS.



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Example (Dafermos):

Consider the following action:

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Consider now the same action, but with the *wrong* sign for the scalar kinetic term:

$$S = \int \mathrm{d}^d x \sqrt{-g}\left[R + (\nabla\phi)^2\right].$$

There is no positivity energy theorem, but Minkowski space is still nonlinearly stable.

Why is AdS unstable?

- AdS acts like a confining finite box. Any generic finite excitation which is added to this box might be expected to explore all configurations consistent with the conserved charges of AdS - including small black holes.

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- A perhaps more convincing intuitive picture: colliding exact plane waves produces singularities - Penrose - '71.

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where $T^{(i)}$ depends on $\{h^{(j \leq i-1)}\}$ and their derivatives and

$$2\tilde{\Delta}_L h_{ab}^{(i)} \equiv -\bar{\nabla}^2 h_{ab}^{(i)} - 2\bar{R}_a{}^c{}_b{}^d h_{cd}^{(i)} - \bar{\nabla}_a \bar{\nabla}_b h^{(i)} + 2\bar{\nabla}_{(a} \bar{\nabla}^c h_{b)c}^{(i)}.$$

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- Any smooth symmetric two-tensor can be expressed as a sum of fundamental building blocks, $\mathcal{T}^{\ell m}_{ab}$, that have definite transformation properties under the $SO(d-1)$ subgroup of AdS.

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- At each order, we can reduce the metric perturbations to 4 gauge invariant functions satisfying (Kodama and Ishibashi '03 for $i = 1$):

$$\Box_2 \Phi^{\alpha,(i)}_{\ell m}(t,r) + V^{(i)}_\ell(r) \Phi^{\alpha,(i)}_{\ell m}(t,r) = \tilde{T}^{\alpha,(i)}_{\ell m}(t,r),$$

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  where $\alpha \in \{c, s\}$ and $\Box_2$ is the w. op. in the (t,r) orbit space.
- Choice of initial data relates $\Phi^{c,(i)}_{\ell m}$ and $\Phi^{s,(i)}_{\ell m}$: 2 PDEs to solve.

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- This is also the choice that conforms with finite energy for the standard definition of "*gravitational energy*".

Linear Perturbations

- At the linear level $(i = 1)$ we can further decompose our perturbations as

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- Because AdS acts like a confining box, only certain frequencies are allowed to propagate

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where p is the radial overtone. These are the so-called normal modes of AdS. The fact that $\omega^2 L^2 > 0$ means that AdS is linearly stable.

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  where p is the radial overtone. These are the so-called normal modes of AdS. The fact that $\omega^2 L^2 > 0$ means that AdS is linearly stable.
- For simplicity, we will take $p = 0$, in which case one finds

$$\Phi^{\alpha,(1),\kappa}(r) = A^{\alpha,(1),\kappa}\frac{r^{\ell+1}}{(r^2+L^2)^{\frac{\ell+1}{2}}},$$

  where $A^{\alpha,(1),\kappa}$ is a normalization constant.

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4. If $\tilde{T}^{\alpha,(i+1)}_{\ell m}(t,r)$ has an harmonic time dependence $\cos(\omega\, t)$, then $\Phi^{\alpha,(i+1)}_{\ell m}(t,r)$ will exhibit the same dependence, EXCEPT when ω agrees with one of the normal frequencies of AdS:

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5. If for a given perturbation one can construct $\Phi^{\alpha,(i)}_{\ell m}$ to any order, without ever introducing a term growing linearly in time, the solution is said to be stable and is unstable otherwise.

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- The structure of the equations indicate that there is only one resonant term at each odd order, and that the amplitude of the growing mode can be set to zero by correcting the frequency.
- One can compute the asymptotic charges to fourth order, and they readily obey to the first order of thermodynamics:

$$E_g = \frac{3J_g}{2L}\left(1 - \frac{4901\, J_g}{7560\pi L^2}\right), \quad \omega_2 = \frac{3}{L}\left(1 - \frac{4901\, J_g}{3780\pi L^2}\right),$$

where we defined ϵ by $J_g = \frac{27}{128}\pi L^2 \epsilon^2$.

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  - Geons are likely to be "*more*" stable than AdS because the normal modes of the Geons correspond to continuous deformations of the AdS normal modes.

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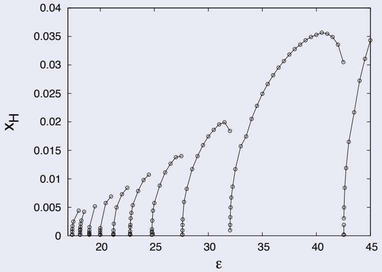
- No matter how small you make the initial amplitude, the curvature at the origin grows and you eventually form a small black hole.



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Consider IIB string theory on $\mathrm{AdS}_5 \times S^5$, with AdS length scale L.

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- If the initial energy is $E_s < E < E_{corr}$, one forms an excited string.
- If $E < E_s$, the cascades stops at frequencies $\omega = E$, and one gets a gas of particles in AdS.

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In 2+1 dimensions, classical turbulence has an inverse energy cascade due to an extra conserved quantity - the enstrophy. Our results indicate that in a strongly coupled quantum theory, there is a standard energy cascade.

Caveat: This intuition comes from solving the Navier Stokes equations in 2+1 dimensions. Because our regime is non-hydro, we don't know how to define enstrophy.

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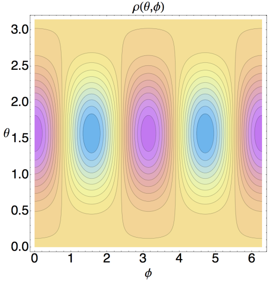

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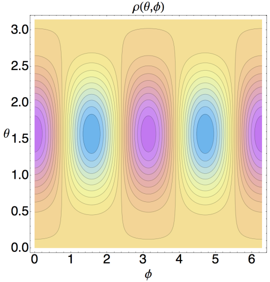
which is timelike near the poles but spacelike near the equator.



Gravitational hairy black holes with a single $U(1)$ - 1/2.

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- The Kerr-AdS is not the unique stationary black hole in AdS.

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- Superposition of modes: superradiance cause them to grow; turbulent instability will cause higher frequency modes to be created.

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2. The black hole exterior might continue to evolve toward higher and higher frequency - black moon?

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Open questions:

- Understand why the energy cascade in 2+1 quantum theory is different from the classical theory.
- Prove a singularity theorem for anti-de Sitter.
- Understand the late time behavior of the superradiance instability.
- Understand the space of CFT states that do not thermalize.