Equivariant cohomology of infinite-dimensional Grassmannian and shifted Schur functions

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1. $\mathcal{H} = L^2(S^1)$: the space of square integrable complex-valued functions on S^1 with normalized Haar measure $dz/2\pi$.
2. $\mathcal{H}_+$: the closed subspace of $\mathcal{H}$ spanned by $\{z^i : i \geq 0\}$.
3. $\mathcal{H}_-$: the closed subspace of $\mathcal{H}$ spanned by $\{z^i : i < 0\}$.
4. $\pi_\pm : \mathcal{H} \to \mathcal{H}_\pm$ the orthogonal projections.

Definition

The infinite dimensional Grassmannian $\mathrm{Gr}(\mathcal{H})$ is the collection of closed subspaces W of $\mathcal{H}$ such that

(1) $\pi_-|_W : W \to \mathcal{H}_-$ is a Fredholm operator (i.e. the kernel is finite-dimensional and the image has finite codimension);

(2) $\pi_+|_W : W \to \mathcal{H}_+$ is a compact operator.

For each $W \in \mathrm{Gr}(\mathcal{H})$, define

$$\operatorname{ind} W = \operatorname{ind} \pi_-|_W = \dim\ker \pi_- - \dim\operatorname{coker} \pi_-.$$

$\operatorname{ind} W$ is called the index of W.

1. G: Lie group
2. EG: a contractible G-space so that G acts freely on it.
3. $BG = EG/G$: the classifying space of G.
4. M: a G-space.
5. $EG \times_G M$: the quotient space of $EG \times G$ modulo the relation $\sim$, where $(pg, q) \sim (p, gq)$ with $p \in EG$, $q \in M$ and $g \in G$.

Definition

The equivariant cohomology of a space M associated with G is defined to be

$$H_G^*(M) = H^*(EG \times_G M).$$

This is a module over $H_G^*(pt) = H^*(BG)$
If $G = S^1$ then the classifying space is the infinite dimensional complex projective space $\mathbb{P}^\infty$ and $H_{S^1}^*(pt) = \mathbb{C}[u]$, where u is a degree 2 element.

The group S^1 acts naturally on $\mathcal{H}$: to every α obeying $|\alpha| = 1$ we assign a map $f(z) \to f(\alpha z)$. This action (we will call it standard action) generates an action of S^1 on Grassmannian . Representing a function on a circle as a Fourier series $f(z) = \sum a_n z^n$, we see that the standard action sends $a_k \to \alpha^k a_k$. One can consider more general actions of S^1 on $\mathcal{H}$ sending $a_k \to \alpha^{n_k} a_k$ where $n_k \in \mathbb{Z}$ is an arbitrary doubly infinite sequence of integers. This action also generates an action of S^1 on Grassmannian.

One can consider the infinite-dimensional torus $\mathbb{T}$ and its action on the Grassmannian. Algebraically the infinite torus $\mathbb{T}$ is the infinite direct product $\prod_{i\in\mathbb{Z}} S^1$. The action of $\mathbb{T}$ on Grassmannian corresponds to the action on $\mathcal{H}$ transforming $a_k \to \alpha_k a_k$, where $(\alpha_k) \in \mathbb{T}$ and $f = \sum_n a_n z^n \in \mathcal{H}$. This action specifies an embedding of $\mathbb{T}$ into the group of unitary transformations of $\mathcal{H}$; the topology of $\mathbb{T}$ is induced by this embedding. One can prove that the equivariant cohomology $H_{\mathbb{T}}(pt)$ (cohomology of the classifying space $B_{\mathbb{T}}$) is isomorphic to the polynomial ring $\mathbb{C}[\mathbf{u}]$ where $\mathbf{u}$ stands for the doubly infinite sequence u_k.

Let $x = (x_n)_{n\in\mathbb{N}}$ be a sequence of variables obeying $x_n = 0$ for $n >> 0$ and $y = (y_i)_{i\in\mathbb{Z}}$ be a doubly infinite sequence of variables. Denote the set of pairs (x, y) by R. Let us consider a function $f(x|y)$ such that its restriction f_n to the subset R_k specified by the condition $x_{k+1} = x_{k+2} = \cdots = 0$ is a polynomial for every $k \in \mathbb{N}$. We say that f is shifted symmetric if f_n symmetric with respect to the variables $x'_i = x_i + y_{-i}$ for $1 \leq i \leq n$. In other words,

$$f'_n(x'_1, \ldots, x'_n|y) = f_n(x'_1 - y_{-1}, \ldots, x'_n - y_{-n}|y)$$

is symmetric with respect to $x' = (x'_1, \ldots, x'_n)$.
If we replace y_j by constant $+\, j$, we obtain the definition of the shifted symmetric functions given by Okounkov and Olshanski. An essentially equivalent notion was introduced by Molev. Instead of R one can consider a set $\tilde{R}$ of pairs (x, y) where $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_i)_{i\in\mathbb{Z}}$ are sequences obeying $x_n = y_{-n}$ for $n >> 0$. Shifted symmetric functions on R correspond to symmetric functions on $\tilde{R}$; this correspondence can be used to relate our approach to Molev's approach. It is obvious that shifted symmetric functions on R constitute a ring; we denote this ring by $\Lambda^*(x\|y)$. This ring is isomorphic to the ring $\Lambda(x\|y)$ of symmetric functions on $\tilde{R}$ considered by Molev.
The equivariant cohomology $H^*_{\mathbb{T}}(\mathrm{Gr}_d(\mathcal{H}))$ is isomorphic to the ring $\Lambda^*(x\|y)$.

Let $x = (x_1, \cdots, x_n)$ be an n-tuple of variables and $y = (y_i)_{i\in\mathbb{Z}}$ be a doubly infinite sequence. Recall that, the double Schur function ${}^n s_\lambda(x_1, \cdots, x_n|y)$ is [1] a symmetric polynomial in $x = (x_1, \cdots, x_n)$ with coefficients in $\mathbb{C}[y]$ defined by

$$ {}^n s_\lambda(x_1, \cdots, x_n|y) = \det\left[(x_i|y)^{\lambda_j+n-j}\right] / \det\left[(x_i|y)^{n-j}\right], $$

where $(x_i|y)^p = \prod_{j=1}^{p}(x_i - y_j)$. Let us introduce the shift operator on $\mathbb{C}[y]$ given by

$$ (\tau y)_i = y_{i-1}, \quad i \in \mathbb{Z}. $$

Then the double Schur function satisfies the generalized Jacobi-Trudi formula :

$$ {}^n s_\lambda(x_1, \cdots, x_n|y) = \det\left[h_{\lambda_i+j-i}(x_1, \cdots, x_n|\tau^{j-1}y)\right]_{i,j=1}^{n}, \qquad (1) $$

where

$$ h_p(x_1, \cdots, x_n|y) = \sum_{1\le i_1\le\cdots\le i_p\le k} (x_{i_1} - y_{i_1})\cdots(x_{i_p} - y_{i_p+p-1}), \quad p \ge 1. $$

[1] We add the index n to the conventional notation s_λ to emphasize that the function depends on n variables x_k.

We define the shifted double Schur function by

$$ {}^n s^*_\lambda(x_1, \cdots, x_n|y) ={}^n s_\lambda(x_1 + y_{-1}, x_2 + y_{-2}, \cdots, x_n + y_{-n}|\tau^{n+1}y). \quad (2) $$

Under the change of variables $x'_i = x_i + y_{-i}$ for $1 \leq i \leq n$, the shifted double Schur function ${}^n s^*_\lambda(x_1, \cdots, x_n|y)$ becomes the double Schur function ${}^n s_\lambda(x'_1, \cdots, x'_n|\tau^{n+1}y)$. Notice that the shifted double Schur function ${}^n s^*_\lambda(x_1, \cdots, x_n|y)$ coincides with the shifted Schur function defined by Okounkov and Olshanski if $y = (y_k)_k$ is the sequence defined by the relation $y_k = \text{constant} + k$ for all k.

One can prove that *the shifted double Schur functions* ${}^n s^*_\lambda(x_1, \cdots, x_n|y)$ *have the following stability property:*
If $l(\lambda) < n$, then

$$ {}^{n+1} s^*_\lambda(x_1, \cdots, x_n, 0|y) = {}^n\, s^*_\lambda(x_1, \cdots, x_n|y). $$

Here $l(\lambda)$ is the length of a partition λ.
Using this property, we can define the shifted double Schur function $s^*_\lambda(x|y)$ depending on infinite number of arguments $x = (x_i)_{i\in\mathbb{N}}$ and $y = (y_j)_{j\in\mathbb{Z}}$ (we assume that only finite number of variables x_i does not vanish). Namely, *we define*

$$ s^*_\lambda(x|y) = \ {}^n s^*_\lambda(x_1, \cdots, x_n|y) $$

where n *is chosen in such a way that* $n > l(\lambda)$ *and* $x_i = 0$ *for* $i > n$.
The shifted double Schur functions $\{s^*_\lambda(x|y)\}$ form a linear basis for $\Lambda^*(x\|y)$ considered as $\mathbb{C}[y]$-module.

The Grassmannian $\mathrm{Gr}(\mathcal{H})$ has a stratification in terms of Schubert cells having finite codimension; it is a disjoint union of $\mathbb{T}$-invariant submanifolds Σ_S labeled by S, where S is a subset of $\mathbb{Z}$ such that the symmetric difference $\mathbb{Z}_- \Delta S$ is a finite set. The Schubert cells Σ_S are in one-to-one correspondence with the $\mathbb{T}$-fixed points $\mathcal{H}_S$, where $\mathcal{H}_S$ is the closed subspace of $\mathcal{H}$ spanned by $\{z^s : s \in S\}$ (the fixed point $\mathcal{H}_S$ is contained in the Schubert cell Σ_S). Instead of a subset S of $\mathbb{Z}$, we can consider a decreasing sequence (s_n) of integers. It is easy to check that $s_n = -n + d$ for $n \gg 0$, where d is the index of $\mathcal{H}_S$. The complex codimension of the Schubert cell Σ_S is given by the formula

$$\Sigma_S = \sum_{i=1}^{\infty} (s_i + i - d).$$

The closure $\bar{\Sigma}_S$ of Σ_S is called the Schubert cycle of characteristic sequence S. It defines a cohomology class in $H^*(\mathrm{Gr}(\mathcal{H}))$ having dimension equal to $2\Sigma_S$. Since the Schubert cycle $\bar{\Sigma}_S$ is $\mathbb{T}$-invariant, it specifies also an element Ω_S^T in $H^*_{\mathbb{T}}(\mathrm{Gr}(\mathcal{H}))$. Denote $\lambda_n = s_n + n - d$ for $n \geq 1$. Then (λ_n) form a partition. Instead of using the sequence S to label the (equivariant) cohomology class Ω_S^T, we use the notation Ω_λ^T. Then the dimension of Ω_λ^T is equal to $2|\lambda|$. Similarly, we denote Σ_S by Σ_λ.

The inclusion map $\iota_S : \{\mathcal{H}_S\} \to \mathrm{Gr}_d(\mathcal{H})$ induces a homomorphism:

$$\iota_S^* : H^*_{S^1}(\mathrm{Gr}_d(\mathcal{H})) \to H^*_{S^1}(\{\mathcal{H}_S\})$$

called the restriction map. Denote by $\delta = (\delta_i)$ the partition corresponding to S, i.e. $\delta_i = s_i + i - d$. Assume that $\lambda = (\lambda_i)$ is a partition such that $l(\lambda) < l(\delta)$.
Then

$$\iota_S^* \Omega_\lambda^T = u^{|\lambda|} s_\lambda^*(\delta_1, \cdots, \delta_i, \cdots).$$

Here s_λ^* *is the shifted Schur function defined by Okounkov and Olshanski.*
Similar statement is true for $\mathbb{T}$ *-equivariant cohomology.*

We can introduce a comultiplication in the ring $H^*_{\mathbb{T}}(\mathrm{Gr}_d(\mathcal{H}))$ using the map

$$\rho : \mathrm{Gr}_0(\mathcal{H}') \times \mathrm{Gr}_0(\mathcal{H}'') \to \mathrm{Gr}_0(\mathcal{H}' \oplus \mathcal{H}'') \tag{3}$$

defined by $(V, W) \mapsto V \oplus W$. Namely, we take $\mathcal{H}' = \mathcal{H}_{even}$ (the subspace spanned by z^{2k}) and $\mathcal{H}'' = \mathcal{H}_{odd}$ (the space spanned by z^{2k+1}). Then $\mathcal{H}' \oplus \mathcal{H}'' = \mathcal{H}$ and the map ρ determines a homomorphism of $\mathbb{T}$- equivariant cohomology

$$H_{\mathbb{T}}(\mathrm{Gr}_0(\mathcal{H})) \to H_{\mathbb{T}}(\mathrm{Gr}_0(\mathcal{H}_{even}) \times \mathrm{Gr}_0(\mathcal{H}_{odd})).$$

It is easy to prove that $H_{\mathbb{T}}(\mathrm{Gr}_0(\mathcal{H}_{even}) \times \mathrm{Gr}_0(\mathcal{H}_{odd}))$ is isomorphic to tensor product of two copies of $H^*_{\mathbb{T}}(\mathrm{Gr}_0(\mathcal{H}))$; we obtain a comultiplication Δ in $H^*_{\mathbb{T}}(\mathrm{Gr}_0(\mathcal{H}))$.
Let us introduce the notion of the k-th shifted power sum function in $\Lambda^*(x\|y)$:

$$p_k(x|y) = \sum_{i=1}^{\infty} \left[(x_i + y_{-i})^k - y_{-i}^k \right]. \tag{4}$$

Here we assume that $x_n = 0$ for $n \gg 0$; hence (4) is a finite sum. We will prove that

$$\Delta p_k = p_k \otimes 1 + 1 \otimes p_k, \quad k \geq 1. \tag{5}$$