Closed String Field Theory and its generalizations

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March 6, 2012

$\mathcal{M}_{g,n}$: moduli space of (compact complex) curves of genus g with n marked points. When $n = 0$, denote $\mathcal{M}_{g,0} = \mathcal{M}_g$.
$\widehat{\mathcal{M}}_{g,n}$: the moduli space of curves of genus g with n marked points and local coordinate systems around each marked point (with n embedded disks).

$\widehat{\mathcal{M}}_n = \bigcup_g \widehat{\mathcal{M}}_{g,n}$ moduli space of (not necessarily connected) complex curves with n disks.
$\widehat{\mathcal{M}}_n \times \widehat{\mathcal{M}}_k \to \widehat{\mathcal{M}}_{n+k}$
disjoint union
$\widehat{\mathcal{M}}_n \to \widehat{\mathcal{M}}_{n-2}$
pasting together two disks
$(S^1)^n$ and S_n act on $\widehat{\mathcal{M}}_n$.
Compatibility
Generalization
MO (modular operad)- collection of topological spaces P_n with actions of S_n and maps $P_n \times P_m \to P_{n+m}$, $P_n \to P_{n-2}$.
EMO (equivariant modular operad)- collection of topological spaces P_n with actions of S_n and $(S^1)^n$ and maps $P_n \times P_m \to P_{n+m}$, $P_n \to P_{n-2}$.

Conformal field theory=collection of maps $\widehat{\mathcal{M}}_n \to H^{\otimes n}$ where H is a vector space equipped with inner product
These maps can be defined up to a factor if the conformal charge $c \neq 0$. They should be compatible with maps $H^{\otimes n} \otimes H^k \to H^{\otimes n+k}$ and $H^{\otimes n} \to H^{\otimes n-2}$.
Algebra over MO = collection of maps $P_n \to H^{\otimes n}$, compatible with maps $H^{\otimes n} \otimes H^k \to H^{\otimes n+k}$ and $H^{\otimes n} \to H^{\otimes n-2}$.
In other words algebra over MO P_n is a homomorphism of MO P_n into standard linear MO $H^{\otimes n}$
Algebra over EMO= algebra over MO +compatibility with action of $(S^1)^n$.

$\mathcal{F}$ -Fock space =representation of the canonical (anti) commutation relations

$$[a_k, a_l^+] = \delta_{kl}, [a_k, a_l] = [a_k^+, a_l^+] = 0,$$

$$[a_k, a_l^+]_+ = \delta_{kl}, [a_k, a_l]_+ = [a_k^+, a_l^+]_+ = 0,$$

with cyclic vector Φ obeying $a_k \Phi = 0$.
Let us consider a space $\mathcal{H}$ of linear combinations $A = \sum(\alpha_k a_k + \beta_k a_k^+)$ with inner product defined as (anti)commutator. For every vector $\Psi \in \mathcal{F}$ we define a subspace $Ann\Psi \subset \mathcal{H}$ as a set of operators $A \in \mathcal{H}$ obeying $A\Psi = 0$. It is easy to check that this subspace is isotropic. By definition isotropic Grassmannian $IGr(\mathcal{H})$ consists of such subspaces $V \subset \mathcal{H}$ that the condition $A\Psi = 0$ for all $A \in V$ specifies the vector Ψ uniquely (up to a factor).

Introducing an inner product in the Fock space $\mathcal{F}$ we obtain a standard linear MO $\mathcal{F}^{\otimes n}$. For appropriate choice of inner product we obtain a structure of MO on isotropic Grassmannians $IGr(\mathcal{H}^n)$. Krichever map specifies an embedding of the MO $\widehat{\mathcal{M}}_n$ into this MO. The linear MO $\mathcal{F}^{\otimes n}$ can be considered as an algebra over the MO $IGr(\mathcal{H}^n)$. Using Krichever map we obtain examples of conformal field theories (free field theories)

Recall that the Krichever map sends a point of $\widehat{\mathcal{M}}_n$ into a point of Sato Grassmannian (into a space of boundary values of differentials holomorphic outside of the embedded disks). The Sato Grassmannian can be considered as a subspace of isotropic Grassmannians.

$\mathcal{D}$= Lie algebra of holomorphic vector fields on the punctured disk. Direct sum $\mathcal{D}^n$ of n copies of $\mathcal{D}$ acts transitively on $\widehat{\mathcal{M}}_n$. This action gives a projective action of $\mathcal{D}$ on the space of states of conformal field theory (a representation of Virasoro).

We extend the space H of states of CFT adding ghosts $b, c, \bar{b}, \bar{c}$ (in other words, we consider the space $\hat{H}$ defined as a tensor product of H and the Fock space of ghosts). The space $\hat{H}$ is equipped with BRST operator Q (with the differential calculating semi-infinite Lie algebra cohomology).

The conformal field theory (algebra $\widehat{\mathcal{M}}_n \to H^{\otimes n}$) can be extended to an algebra $\Pi T\widehat{\mathcal{M}}_n \to \hat{H}^{\otimes n}$. The differential on $\Pi T\widehat{\mathcal{M}}_n$ is compatible with BRST operator Q on $\hat{H}$.

Here ΠTX stands for the tangent bundle with reversed parity, functions on ΠTX are differential forms on X. The extension is given by $\hat{H}^{\otimes n}$-valued differential forms $\Omega_n = \sum_k b(v_1)\cdots b(v_k)\Sigma_C$ where Σ_C stands for the point of $H^{\otimes n}$ corresponding to $C \in \widehat{\mathcal{M}}_n$ and $b(v)$ corresponds to $v \in \mathcal{D}^n$. We assume that the central charge of extended theory vanishes (this means that $c = 26$.)

The string state $A \in \hat{H}$ obeys

$$l_- A = 0, b_- A = 0$$

where l_- stands for the generator of circle action and $l_- = [Q, b_-]_+$. We denote the space of such states by H'. Functions on H' are called string fields.
The operator Q acts on H' ; its cohomology is identified with with physical string states.
The string amplitudes are defined as integrals of differential forms

$$< A_1 \otimes \cdots \otimes A_n | \Omega_n >$$

over "fundamental cycle" in moduli space (coming from Deligne-Mumford compactification)

String field theory is specified by an action functional S defined on H' and obeying quantum master equation

$$\Delta e^{\frac{S}{\hbar}} = 0$$

Here Δ stands for the BV operator (odd Laplacian). It is well defined because H' can be considered as an odd symplectic manifold equipped with a volume element.
Recall that an odd symplectic manifold is specified by a closed odd non-degenerate two-form and Δ is given by the formula

$$\Delta = \omega^{AB} \frac{\partial^2}{\partial z^A \partial z^B}.$$

Functions on such a manifold form a BV-algebra (a supercommutative algebra equipped with an odd second order differential operator Δ obeying $\Delta^2 = 0$).

Sen-Zwiebach BV algebra of singular chains.
Operation Δ (BV operator) on S_n-invariant singular chains in $\widehat{\mathcal{M}}_n/(S^1)^n$ (equivariant singular chains in $\widehat{\mathcal{M}}_n$) transforms k -dimensional equivariant chain on $\widehat{\mathcal{M}}_n$ into $(k+1)$-dimensional chain on $\widehat{\mathcal{M}}_{n-2}$

$$(\partial + \hbar\Delta)e^{\frac{\nu}{\hbar}} = 0$$

$$\nu = \sum \hbar^g \nu_g$$

Under certain conditions one can prove that the solution of this equation is essentially unique.

String background (e.g. conformal field theory of critical dimension +ghosts) specifies a homomorphism of Sen-Zwiebach BV algebra into the BV algebra of string fields: we integrate the form

$$< A_1 \otimes \cdots \otimes A_n | \Omega_n >=< A_1 \otimes \cdots \otimes A_n | b(v_1) \cdots b(v_k) \Sigma_C >$$

The solution ν of BV master equation on moduli spaces gives a solution to the master equation in the space of string fields . More precisely, it gives a functional S_{int} obeyng

$$(Q + \Delta) e^{\frac{S_{int}}{\hbar}} = 0$$

To get a solution to the master equation we add to S_{int} a quadratic term (the "free action").
String background can be specified by topological conformal theory.
Topological string. Costello.

BV algebra of singular chains can be constructed for any EMO. For Grassmannian EMO the master equation can be solved. (The homology of the operator $\partial + \Delta$ in the spaces of equivariant chains coincides with the conventional equivariant homology).
Problem: image of solution to the master equation on singular chains in moduli spaces in Grassmannians.
Bosonic string can be embedded into "Grassmannian" string