Gauge Symmetry and Dynamics in Anisotropic Gravity

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Reproduce the low-energy GR spectrum
in an anisotropic model of gravity.

Anisotropic Gravity?

Idea Behind Anisotropic Gravity

The original idea behind gravity at a Lifshitz point was to combine gravity with anisotropic scaling.

Condensed matter has many systems characterized by scaling symmetries that are anisotropic in space and time:

$$(t, \vec{x}) \mapsto (\lambda^z t, \lambda \vec{x}). \tag{1}$$

z is the dynamical critical exponent.

Theories of this type have some useful properties, including improving the high-energy behavior of the propagator.

This can render relativistically non-renormalizable interactions renormalizable.

Why Anisotropic Gravity?

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The last few decades have given new reasons for investigating general theories of gravity.

- Power-counting renormalizability raises the possibility that these theories could have a UV completion in weakly-coupled field theory
- Phenomenological models: UV completion of the Standard Model? (Speed of light problem.)
- Membrane theories
- May arise as gravity duals to some theories as in AdS/CFT, especially for non-relativistic field theories
- And others.

Anisotropy in Scalar Field Theory

Many rich features are present in the simplest example: the Lifshitz scalar

$$S = \frac{1}{2} \int dt d^D x \left(\dot{\phi}^2 - (\Delta \phi)^2 . \right) \tag{2}$$

Dimension of ϕ is

$$[\phi] = \frac{D-2}{2} \tag{3}$$

Critical dimension is $D = 2$ with an infinite number of renormalizable couplings:

marginal—non-linear sigma models
relevant—linear sigma models

Applying Anisotropy to Gravity

Apply the idea to gravity:

The starting point is to write the action as the difference $S = S_K - S_V$ of a kinetic and a potential term:

$$S_K = \frac{1}{2\kappa^2} \int dt\, d^D x \sqrt{g}\, \dot{g}_{ij} G^{ijk\ell} \dot{g}_{k\ell} \tag{4}$$

with (generalized) De Witt metric $G^{ijk\ell} = g^{ik} g^{j\ell} - \lambda g^{ij} g^{k\ell}$ and S_V the contribution from the potential.

In GR the potential is just the Ricci scalar, but we allow anisotropic scaling:

$$S_V = \frac{1}{\kappa^2} \int dt\, d^D x \sqrt{g}\, V(R_{ijk\ell}) \tag{5}$$

where V now involves $2z$ derivatives rather than just 2 derivatives.

Symmetries

Theory is invariant only under spatial reparametrizations $\vec{x}' = \vec{g}(\vec{x})$ that are independent of time.

Introduce local symmetries by adding lapse (N) and shift (N_i) variables as in the ADM decomposition. Make the replacements:

$$\dot{g}_{ij} \mapsto K_{ij} = (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)/N \qquad \sqrt{g} \mapsto N\sqrt{g} \tag{6}$$

The resulting theory has D local symmetries, one fewer than in general relativity.

Geometry

A manifold with foliation is a manifold whose charts have a preferred slicing by hypersurfaces, and whose transition functions preserve that slicing.

In gravity at a Lifshitz point, the geometric starting point is a spacetime manifold with a preferred foliation $\mathcal{F}$ by spatial slices.

Resulting symmetry group is the foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}$, which preserves the foliation $\mathcal{F}$.

Symmetry transformations thus have the form

$$t \mapsto f(t) \qquad \vec{x} \mapsto \vec{g}(t, \vec{x}).$$

N can either be a function of space and time ("non-projectable"), or a function of time only ("projectable").

Minimal case: $N = N(t)$ is a function of time only.

Simple Example: $z = 2$ Gravity in $D + 1$ Dimensions

A simple example has $z = 2$ scaling:

$$S_V = \frac{1}{2\kappa^2} \int dt\, d^D x \left(\alpha R^{ijk\ell} R_{ijk\ell} + \beta R^{ij} R_{ij} + \gamma R^2 \right) \qquad (7)$$

For $z = 2$, the coupling constants are dimensionless when $D = 2$.

In this dimension the theory is power-counting renormalizable.

The dimension of κ^2 for arbitrary z is

$$[\kappa^2] = z - D \qquad (8)$$

The critical dimension is $D = z$.

Infrared Deformations

The Lifshitz scalar can be deformed by relevant operators:

$$S = \frac{1}{2}\int dt\, d^D x \left(\dot{\phi}^2 - (\Delta\phi)^2 - c^2(\nabla\phi)^2 - m^2\phi^2 - \cdots\right) \qquad (9)$$

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In the infrared the relevant operator dominates and the theory flows to a relativistic scalar with limiting speed c:

$$S = \frac{1}{2}\int dt\, d^D x \left(\dot{\phi}^2 - c^2(\nabla\phi)^2 - m^2\phi^2 - \cdots\right) \quad (10)$$

Lorentz invariance emerges generically as an accidental symmetry in the infrared even with a mass term or interactions.

Behavior in the Infrared

The relevant operators in the gravity theory include factors of the form

$$S_{V,IR} \sim \frac{\mu}{2\kappa^2} \int dt\, d^D x \sqrt{g}(R - 2\Lambda). \tag{11}$$

Combining R with the kinetic term and using the appropriate definition of c gives (for $\lambda = 1$) the action of GR.

Contribution to dispersion relation is $\sim k^2$.

Possibility of approaching GR in the infrared?

Infrared Deformations

The theory cannot match GR, however:

- Need $\lambda = 1$
- $N = N(t)$ rather than $N(t, \vec{x})$
- Gauge symmetries do not match

Losing one symmetry reduces the number of constraints in our theory and we expect end up with an extra degree of freedom.

In fact the crucial difference is the existence of an extra propagating scalar degree of freedom (essentially the trace of metric perturbations).

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Problem: The scalar dispersion relation has the wrong sign in the infrared ($\omega^2 \sim -k^2$). So there is an instability.

Possibility 1: Modulated Phases

One way to try to solve the problem is to let the theory go where it wants and see where it ends up.

Situation has an analogue in the Lifshitz scalar theory:

$$S = \frac{1}{2} \int dt\, d^D x \left(\dot{\phi}^2 - (\Delta\phi)^2 - c^2 (\nabla\phi)^2 - \cdots \right) \tag{12}$$

This theory describes the tricritical point of water at the intersection of three phases: ordered, disordered, and spatially modulated.

If the value of $c^2 = -\mu^2$ is negative then the time-independent configurations must satisfy $\mu^2\kappa^2\Delta\phi + \Delta^2\phi = 0$ and stable configurations have $\phi \sim \sin(\mu\kappa x)$.

Unstable $\phi = 0$ state decays into a stable modulated phase.

Phases of Gravity

[in progress, w/Petr Hořava, Patrick Zulkowski, Kevin Grosvenor]

Varying the parameters in the action leads to phases of gravity with distinct behavior. (c.f. Ambørn et al., . . .)

For example, when spatial slices have $k = 1$ we find

- solutions similar to de Sitter,
- solutions oscillating in time, and
- the Einstein static universe along the transition line ("detailed balance").

In more involved constructions we also find phases of gravity that are spatially modulated and stationary in time.

Possibility 2: Eliminate the Scalar Mode

A second is to try to eliminate the mode entirely.

The presence of the extra mode is related to

- $N(t)$ is a function of time only
- Fewer gauge symmetries in our model

The most natural way to resolve this is to introduce a new non-relativistic form of general covariance: expand the gauge group by some form of local geometric symmetry.

Finding a New Symmetry

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- Start by from a peculiar result: in the linearized limit, the theory is invariant under the transformation $\delta N_i(t, \vec{x}) = \partial_i f(\vec{x})$ precisely when $\lambda = 1$ (the GR value)
- Demanding this to be a symmetry therefore fixes λ at the GR value
- Not a gauge symmetry (in the usual sense) — it does not depend on time

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$$L_{\mathrm{photon}} \sim \frac{1}{2}\dot{A}_i^2 - \frac{1}{4}F_{ij}F^{ij} - A_j J^i$$

This action has well-defined time evolution, but it only gives EM when supplemented by the initial condition requirement $\vec{\nabla} \cdot \vec{E} \sim \rho$ (Gauss constraint).

Also has a residual gauge symmetry: $\delta A_i(t, \vec{x}) = \nabla_i f(\vec{x})$; note that f depends only on space and not on time.

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Since linearized anisotropic gravity has such a time-independent—but spatially local—symmetry, we can try to gauge it in time in such a way that the action is the gauge-fixed form of another theory.

Gauging the Abelian Global Symmetry

We want a gauge field for the new abelian symmetry.

A field with the appropriate transformation can be derived from the non-relativistic contraction of the diffeomorphism algebra: consider the $1/c$ expansion of the tt component of the metric,

$$g_{00} = -N^2 + \frac{1}{c^2} 2NA(t, \vec{x}) + \cdots .$$

Subleading correction to all fields under time reparametrizations is:

$$\delta A = \dot{\alpha} - N^i \partial_i \alpha \qquad \delta N_i = \partial_i \alpha \qquad \delta g_{ij} = 0$$

Try to gauge the symmetry by introducing a field with this transformation property.

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Gauge symmetry + extra constraint eliminates a degree of freedom

However there is an obstruction to extending this to the nonlinear theory:

$$\delta S_\alpha \sim \int dt dx \sqrt{g} \, (R^{ij} - \frac{1}{2} R g^{ij}) K_{ij} \cdot \alpha$$

Eliminating the Obstruction (1)

The obstruction vanishes in several cases of interest:

- $D = 2$, where the Einstein tensor vanishes identically
- Abelianized gravity need not be non-interacting; renormalizable anisotropic interacting theories of abelianized gravity with this symmetry exist
- We can add the subleading terms A_{ij} in the $1/c$ expansion of g_{ij}; this allows closure of the algebra, but kills all propagating modes, rendering the theory topological.

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If we want dynamical gravity for $D > 2$ we need a different solution.

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The obstruction can now be cancelled by terms of the form

$$S \sim \int dtdxN\sqrt{g}(R^{ij} - \frac{1}{2}Rg^{ij} - \Omega g^{ij})(K_{ij}\nu - \nabla_i\nu\nabla_j\nu)$$

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Promotes abelian symmetry to a local symmetry.

Next: Check the spectrum.

Spectrum Analysis

Can check the number of degrees of freedom about a general background using the Hamiltonian formalism:

- Varying A generates a first-class constraint; ν gives a second class constraint
- Gauge-fixing $+$ first-class constraint
  $\Longrightarrow$ dimension of physical phase space reduced by 2 from original theory

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  $\implies$ dimension of physical phase space reduced by 2 from original theory
  $\implies$ eliminates the scalar degree of freedom
- We are left with only a transverse traceless graviton with a quadratic dispersion in the IR
  $\implies$ spectrum matches GR

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In addition to a matching spectrum, the static solutions (at $\Lambda = \Omega = 0$) also coincide in the IR, including Schwarzschild:

- Static solution $\implies K_{ij} = 0$
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- Same as for GR, provided the GR lapse function $\mathcal{N}$ is matched with $N - A \implies$ it has the same static solutions.

Outlook

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However, there are important questions remaining:

- Role of the Newton prepotential
- Coupling to matter
- Speed of light problem
- Cosmology? Dark matter?
- etc.

End.