115 Homework 1 Solutions

Due Friday October 8

Question 1 Prove that $\sqrt{5}$ is irrational.

Proof: (Similar to the book, pg. 7)

Suppose that $\sqrt{5}$ were rational. Then there would exist positve integers p and q with $\sqrt{5}$ =p/q. Consequently, the set S = $\{k\sqrt{5} \mid k$ and $k\sqrt{5}$ are positve integers $\}$ is nonempty since p = $q\sqrt{5}$. Therefore, by the well-ordering property, S has a smallest element, $s = t\sqrt{5}$. Since $s\sqrt{5}$ and s are both integers, $s\sqrt{5} - 2s$ = $(s-2t)\sqrt{5}$ must also be an integer. Furthermore, it is positve, since $s\sqrt{5} - 2s = s(\sqrt{5}-2)$ and $\sqrt{5}-2 > 0$. It, $s\sqrt{5}-2s$, is less than s since $s\sqrt{5}-2s = (\sqrt{5}-2)s$ and $\sqrt{5} - 2 < 1$. This contradicts the choice of s as the smallest integer in S. It follows that $\sqrt{5}$ is irrational.■

Here is another, less detailed, proof, that does not use the well-ordering property. Suppose that $\sqrt{5}$ were rational. Then there would exist positve integers p and q with $\sqrt{5}$ =p/q. We then have $5 = p^2/q^2$ or $5q^2 = p^2$. From the fundamental theorem of arithmetic we have $p = p_1^{a_1}p_2^{a_2}\ldots p_m^{a_m}$ and $q = q_1^{b_1}q_2^{b_2}\ldots q_n^{b_n}$. Our equation now becomes $5q_1^{2b_1}q_2^{2b_2}\ldots q_n^{2b_n} = p_1^{2a_1}p_2^{2a_2}\ldots p_m^{2a_m}$. Now according to the right hand side of the equation every prime factor must occur an even number of times, but looking at the left hand side clearly the factor 5 occurs an odd number of times. We have reached a contradiction and it follows that $\sqrt{5}$ is irrational.■

Question 2 (Rosen 1.2.8) Use mathematical induction to show

$$1^3 + 2^3 + 3^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$$

.

Proof:

First the base case:

$$\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1^3$$

.

Now the inductive case: Suppose

$$1^3 + 2^3 + 3^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$$

Then for $n+1$ we have $1^3 + 2^3 + 3^3 + \cdots n^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$ by our inductive hypothesis. Then performing some simplification on the right hand side we have:

$$\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n^2+2n+1)}{4} + n^3 + 3n^2 + 3n + 1 =$$

$$\frac{n^4 + 2n^3 + n^2}{4} + \frac{4n^3 + 12n^2 + 12n + 4}{4} = \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} =$$

$$\frac{(n^2+4n+4)(n^2+2n+1)}{4} = \frac{(n+2)^2(n+1)^2}{4}$$

Thus we have $1^3 + 2^3 + 3^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers n.■

Question 3 (Rosen 1.3.4) Derive a simple formula for sums of even index Fibonacci numbers

$$f_2 + f_4 + \cdots + f_{2n}\,.$$

Let's look at the first few cases before we try to prove it:

$$f_2 = 1$$

$$f_2 + f_4 = 1 + 3 = 4$$

$$f_2 + f_4 + f_6 = 1 + 3 + 8 = 12$$

$$f_2 + f_4 + f_6 + f_8 = 1 + 3 + 8 + 21 = 33$$

Looking at the results there is a (possible) pattern emerging, namely

$$f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1.$$

Proof: (Use induction) First the base case: Obviously $f_2 = 1 = 2 - 1 = f_3 - 1$

Now the inductive case: Suppose

$$f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1.$$

Now for $n+1$

$$f_2 + f_4 + \cdots + f_{2n} + f_{2(n+1)} \ = f_{2n+1} - 1 + f_{2n+2} =$$

$$f_{2n+3} - 1 = f_{2(n+1)+1}$$

Thus we have $f_2 + f_4 + \cdots + f_{2n} \ = f_{2n+1} - 1$ for all natural numbers n.■

Question 4 (Rosen 1.4.22) Show that

$$[x] + [x + \frac{1}{2}] = [2x]$$

for any real number x.

Proof: Let $x = [x] + r$ where $0 \leq r < 1$ We proceed by considering two cases:

case i: Suppose $r < 1/2$. Then $x+1/2 = [x]+r+1/2 < [x]+1$ since $r+1/2 < 1$. It follows that $[x + 1/2] = [x]$. Also $2x = 2[x] + 2r < 2[x] + 1$ since $2r < 1$. Hence $[2x] = 2[x]$ It follows that $[x] + [x + 1/2] = [2x]$.

case ii: Suppose $1/2 \leq r < 1$. Then $[x] + 1 \leq x + (r + 1/2) < [x] + 2$, so that $[x+1/2] = [x]+1$. Also $2[x]+1 \leq 2[x]+2r = 2([x]+r) = 2r < 2[x]+2$ sp that $[2x] = 2[x]+1$. It follows that $[x]+[x+1/2] = [x]+[x]+1 = 2[x]+1 = 2[x]$.■