115 Homework 2

Due Friday October 15

Question 1 Find primes that are the difference of cubes of two integers. Can you place conditions on such pairs of integers?

First we must have $a > b$ for $a^3 - b^3$ to even have a chance at being prime. Lets find some primes that are the difference of cubes.

$$2^3 - 1^3 = 7, 3^3 - 2^3 = 19, 4^3 - 3^3 = 37, 5^3 - 4^3 = 61$$

Now we can try to set a condition on the integers a and b in $a^3 - b^3$. If we factor $a^3 - b^3$ we get

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

For this number to be prime exactly one it's the factors must be prime. Clearly $a^2 + ab + b^2 \neq 1$ unless $a = 1$ and $b = 0$, but this results in $a^3 - b^3 = 1$, not a prime. Thus we must set $a - b = 1$ or $a = b + 1$. This is a necessary but not sufficient condition for $a^3 - b^3$ to be prime.

Question 2 (Rosen 3.1.4) Find all primes less than 200 using Erastosthenes' sieve.

The Prime numbers less than 200 are:

$$3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61,$$

$$67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 113, 127, 131, 137,$$

$$139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199.$$

Question 3 (Rosen 3.1.6) Show that no integer of the form $n^3 + 1$ is prime other than $2 = 1^3 + 1$.

Note that n must be a positive integer as $n^3 + 1$ is 1 for $n = 0$ and negative for $n < 0$. Now $n^3 + 1 = (n + 1)(n^2 - n + 1)$ and $n^3 + 1$ is not prime unless one of these two factors is one. Clearly $n + 1 \neq 1$ since n is a positive integer, so it must be true that $n^2 - n + 1 = 1 \rightarrow n = 1$ Thus $n^3 + 1$ is prime iff $n = 1 \rightarrow n^3 + 1 = 2$ is the only prime of the form $n^3 + 1$.

Question 4 (Rosen 3.1.12) Find:

- The smallest 5 consecutive composite integers.
- 1,000,000 consecutive composite integers.

- The smallest 5 consecutive composite integers can be found by finding the first two odd composite numbers, namely 25 and 27. Thus the smallest 5 consecutive numbers are $24, 25, 26, 27, 28$.
- $1000001! + j$ is divisible by j for all $j = 2, 3, \ldots, 1000001$ which gives one million consecutive composite integers.

Question 5 (Rosen 3.1.20) What is the least positive integer x such that $x^2 - x + 41$ is composite?

Brute force: $f(0) = 41, f(1) = 41, f(2) = 43, f(4) = 47, f(5) = 53, f(6) = 71, f(7) = 83, f(8) = 83, f(9) = 113, f(10) = 131, f(11) = 151, f(12) = 173, f(13) = 197, f(14) = 223, f(15) = 251, f(16) = 281, f(17) = 313, f(18) = 347, f(19) = 383, f(20) = 421, f(21) = 461, f(22) = 503, f(23) = 547, f(24) = 593, f(25) = 641, f(26) = 691, f(27) = 743, f(28) = 797, f(29) = 853, f(30) = 911, f(31) = 971, f(32) = 1033, f(33) = 1097, f(34) = 1163, f(35) = 1231, f(36) = 1301, f(37) = 1373, f(38) = 1447, f(39) = 1523, f(40) = 1601$ are all primes, but $f(41) = 41^2 - 41 + 41 = 41^2$ is composite.

Question 6 (Rosen 3.2.8) Show that if integers a and b obey $(a, b) = 1$, then $(a + b, a - b) = 1$ or 2.

Suppose that $d \mid (a + b)$ and $d \mid (a - b)$. Then $d \mid ((a + b) + (a - b)) = 2a$ and $d \mid ((a+b) - (a-b)) = 2b$. Note that $(2a, 2b) = 2(a, b) = 2$. Since d is a common divisor of $2a$ and $2b$ it follows that $d \mid 2$. Hence either $d = 1$ or $d = 2$. Moreover, if one of a and b is even and the other odd, then both $a + b$ and $a - b$ are odd, so that $(a + b, a - b) = 1$. If both a and b are odd then both $a + b$ and $a - b$ are even, so that $(a + b, a - b) = 2$.