115 Homework 4 Solutions

Due Friday October 29

Question 1 Definition: if p is prime and $p^a|n$ while $p^{a+1} \nmid n$ we say that p^a divides n *exactly* and write $p||n$. (Rosen 3.4.12) Show that $p^a||m \Rightarrow p^{ka}||m^k$.

Solution If $p^a||m$ then $m = p^a n$ where $p \nmid n$. Then $p \nmid n^k$ and we have $m^k = p^{ka}n^k$ and we see that $p^{ka}||m^k$

Question 2 (Rosen 3.4.17,18) Find all $n \in \mathbb{N}$ such that $n!$ ends with 74 zeros (in base ten). Show $n!$ never ends with 153, 154 or 155 zeros.

Solution i) Suppose that $n!$ ends with exactly 74 zeroes. Then $5^{75} \cdot 2^{74} = 10^{74}|n!$. Since there are more multiples of 2 than 5 in $1, 2, \ldots, n$ we need only concern ourselves with the fact that $5^{74}||n!$. Thus(by exercise 3.4.12) we need to find an n such that $74 = [n/5] + [n/5^2] + [n/5^3] + \cdots$. By direct calculation,

$$[300/5] + [300/5^2] + [300/5^3] = 60 + 32 + 2 = 74$$

It follows that $300!$, $301!$, $302!$, and $304!$ all end with exactly 74 zeroes.

ii) The number of zeroes at the end of $n!$ equals the number of 5's in the prime factorization of $n!$. This is clearly an increasing function of n. There are $\sum_{j=1}^{3}[624/5^j] = 124 + 24 + 4 = 152$ zeroes at the end of the expansion of $624!$. However, since 5^4 divides 625, we see that there are $152+4 = 156$ zeroes at the end of the expansion of $625!$. It follows that there cannot be 153,154, or 155 zeroes at the end of the expansion of $n!$.

Question 3 (Rosen 3.5.4 a,c,e) Use Fermat's method to factor 8051, 46009 and 3,200,399.

Solution a) The smallest square greater than 8051 is $90^2 = 8100$. We see that $90^2 - 8051 = 49 = 7^2$, so that $8051 = 90^2 - 7^2 = (90+7)(90-7) = 97 \cdot 83$.

c) The smallest square greater than 10897 is 105^2. But the smallest integer a such that $a^2 - 10897$ is a square is $a = 329$. So we get that $329^2 - 10897 = 97344 = 312^2$. Then $10897 = (329 - 312)(329 + 312) = 17 \cdot 641$.

e) The smallest square greater than 3200399 is 1789^2. But the smallest integer a such that $a^2 - 3200399$ is a square is $a = 1800$. So we get that $1800^2 - 3200399 = 39601 = 199^2$. Then $3200399 = (1800 - 199)(1800 + 199) = 1601 \cdot 1999$.

Question 4 (Rosen 3.5.20) Find all primes $2^{2^n} + 5$ where $n + 1 \in \mathbb{N}$.

Solution We have $2^{2^0} + 5 = 7$. This is the only prime of the form 2^{2^n} since $2^{2^n} + 5 \equiv (2-3)^{2^n} + 5 \equiv (-1)^{2^n} + 5 \equiv 1 + 5 \equiv 0(\text{mod}3)$ when $n \geq 1$. Alternatively, notice that $2^{2^{n+1}} - 2^{2^n} = (2^{2^n})^2 - 2^{2^n} = 2^{2^n}\left(2^{2^n} - 1\right)$. But it is easy to show $3|2^{2^n} - 1$ by induction because $2^{2^n} - 1 = (2^{2^{n-1}} - 1)(2^{2^{n-1}} + 1)$.

Question 5 (Rosen 3.6.14) A piggy bank contains \$2 made from 24 coins which are nickels, dimes and quarters. What combinations of coins are possible?

Solution Suppose that there are x nickels y dimes and z quarters. Since there are 24 coins in the piggy bank, we know that $x + y + z = 24$. Since there are two dollars in the bank we know that $5x + 10y + 25z = 200$. Multiplying the first equation by 5 and subtracting it from the second yields $5y + 20z = 80$. Dividing both sides by 5 give $y + 4z = 16$. The solutions to the linear diophantine equation are $y = 16 - 4t, z = t$ where t is a positive integer. There are 5 nonnegative solutions for $0 \leq t \leq 4$. We have $y = 16$ and $z = 0$ which gives $x = 8$, $y = 12$ and $z = 1$ which gives $x = 11$,$y = 8$ and $z = 2$ which gives $x = 14$,$y = 4$ and $z = 3$ which gives $x = 17$, $y = 0$ and $z = 4$ which gives $x = 20$. Hence the solutions are 8 nickels, 16 dimes, and 0 quarters; 11 nickels, 12 dimes, and 1 quarter; 14 nickels, 8 dimes, and 2 quarters; 17 nickels, 4 dimes, and 3 quarters; and 20 nickels, 0 dimes, and 4 quarters.