115 Homework 9

Due Friday December 3

Question 1 (Rosen 7.2.8,9,10,11) Which positive integers have exactly (i) two positive divisors, (ii) three positive divisors and (iii) four positive divisors? What is the product of positive divisors for any $\in \mathbb{N}$? (Why?)

Solution (8) The positive integers with exactly two divisors are prime.

(9) The positive integers with exactly three prime divisors are those integers of the form p^2 where p is prime. We see this using the formula given in Theorem 6.8. We have $\tau(p_1^{a_1} \cdots p_t^{a_t}) = (a_1+1)(a_2+1)\cdots(a_n+1)$. Since the terms on the right-hand side are all at least 2, this product can equal 3 if and oly if there is exactly one term on the right-hand side that is equal to 3.

(10) We need $\tau(n) = 4$. If $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ then $\tau(n) = (a_1+1)(a_2+1)\cdots(a_s+1) = 4$, so there are two possibilities. Either $(a_1+1) = 4$ or $(a_1+1) = (a_2+1) = 2$. In the first case $a_1 = 3$ and $n = p^3$. In the second case $a_1 = a_2 = 1$ and $n = p_1 p_2$.

(11) We first suppose that n is not a perfect square. Then the divisors of n come in pairs with the product n, that is, when d is a divisor, so is n/d and conversely. Since there are $\tau(n)/2$ such pairs, the product of all the divisors is $n^{\tau(n)/2}$. Now suppuse that n is a perfect square. Then there are $(\tau(n)-1)/2$ pairs with the product n and the extra divisor $\sqrt{n}$. Hence the product of all the divisors of n is $n^{(\tau(n)-1)/2} \cdot n^{1/2} = n^{\tau(n)/2}$.

Question 2 (Rosen 7.3.11,13) Let $n \in \mathbb{N}$. We call n abundant if $\sigma(n) > 2n$ and deficient if $\sigma(n) < 2n$. Show there are infinitely many deficient numbers as well as infinitely many odd abundant ones.

Solution See the solutions for section 7.3 page 585 in the textbook.

Question 3 (Rosen 7.4.10) Let $n \in \mathbb{N}$. Show $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$.

Solution Since $n, n+1, n+2, n+3$ form a complete residue system modulo 4, one of them is divisible by 4, and so not square free. Therefore one of the factors in $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3)$ is 0, making the product 0.