185A Homework 1

Question 1 Verify the following properties for $z, w \in \mathbb{C}$:

1. $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$
2. $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$
3. $z = \overline{z}$ if and only if $z \in \mathbb{R}$ is purely real.
   $z = -\overline{z}$ if and only if $z \in i\mathbb{R}$ is purely imaginary.
   Find the maximal subset of $\mathbb{C}$ which is left fixed (as a set and then also pointwise) by complex conjugation.

Question 2 Let $\langle z, w\rangle \equiv (\overline{z}w + \overline{w}z)/2$. Prove the following identities:

1. $\langle z, w\rangle^2 + \langle iz, w\rangle^2 = |z|^2|w|^2$
2. $|\langle z, w\rangle| \leq |z| \cdot |w|$ (Cauchy-Schwartz inequality)
3. $|z+w|^2 = |z|^2 + |w|^2 + 2\langle z, w\rangle$ (cosine law)
4. $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$
5. $|z|^2 = \langle z, z\rangle$
6. $|zw| = |z| \cdot |w|$
7. $|z+w| \leq |z| + |w|$ (triangle inequality)
8. $|\frac{z}{w}| = \frac{|z|}{|w|}$.

Question 3 Define the function $\tilde{e}(\theta) = \cos(\theta) + i\sin(\theta)$. Show that

$$\frac{1}{\tilde{e}(\theta)} = \tilde{e}(-\theta)$$

and use this fact to extend de Moivre's theorem to all $n \in \mathbb{Z}$.
Question 4 Prove Lagrange's trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$

where we assume that $\sin\theta/2 \neq 0$.

Question 5 Let $x, y, u, v \in \mathbb{R}$ and study matrices of the form

$$q = \begin{pmatrix} x & v & u & -y \\ -v & x & y & u \\ -u & -y & x & -v \\ y & -u & v & x \end{pmatrix}$$

Do they form a field? (If not, characterize their failure to do so.) [*Hint; Look up the "quaternions"*.]