185A Homework 4

Question 1 Let $f : G \to \mathbb{C}$ be a continuous function on an open set $G \subset \mathbb{C}$ and let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve in G.

(a) Find a counterexample demonstrating that the inequality

$$\left|\int_\gamma f(z)dz\right| \leq \int_\gamma |f(z)|dz$$

no longer makes sense for integrals along a curve γ.

(b) Show that

$$\left|\int_\gamma f(z)\right| \leq \int_\gamma |f(z)||dz|$$

where the latter is defined by

$$\int_\gamma |f(z)||dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|dt.$$

Question 2 Deduce from Question 1 that

$$\left|\int_\gamma f\right| \leq M\ell(\gamma)$$

where $M \geq 0$ is a real constant such that $|f(z)| \leq M$ for all points z on γ and

$$\ell(\gamma) = \int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$$

is the length of the curve.

Question 3 Let γ be the arc of the circle $|z| = 2$ in the first quadrant $(x, y > 0)$. Establish the inequality

$$\left|\int_\gamma \frac{dz}{1+z^2}\right| \leq \frac{\pi}{3}$$

without performing the integral explicitly.

Question 4 Compute $\int_\gamma f(z)\,dz$ for the following

(a) $f(z) = -y^2 + x^2 - 2ixy$ and γ the straight line from 0 to $-1 - i$.
(b) $f(z) = (2 + z)/z$ and γ the semi-circle $z = \exp(i\theta)$, $0 \leq \theta \leq \pi$.
(c) $f(z) = 1/z$ and γ any path in the right half plane $\mathrm{Re}(z) \geq 0$ beginnng at $-i$, ending at i, avoiding the origin.

Question 5 Let f, g be continuous functions, c_1, c_2 complex constants and $\gamma, \gamma_1, \gamma_2$ piecewise smooth curves. Show that

(a) $$\int_\gamma (c_1 f + c_2 g) = c_1 \int_\gamma f + c_2 \int_\gamma g$$

(b) $$\int_{-\gamma} f = -\int_\gamma f$$

(c) $$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f\,,$$

where $\gamma_1 + \gamma_2$ denotes concatenation of curves.