185A Homework 5

Question 1 Compute $\oint_\Gamma dz\, \pi \exp(\pi \bar{z})$ where Γ is the square with vertices at 0, 1, $1+i$ and i traversed in the positive sense.

Question 2 The Maximum Modulus Principle: Harmonic functions cannot take local maxima or minima. A similar statement holds for the modulus of an analytic function $f(z) \neq$ constant. In this exercise we show that the modulus of an analytic function cannot take a maximum within its domain of analyticity D.

(i) Proceed by contradiction and assume $|f(z_0)|$ is a local maximum for $z_0 \in D$. Write a formula for $f(z_0)$ in terms of an integral around a small circle, center z_0.

(ii) Rewrite this integral as a line integral with parameter θ by calling $z = z_0 + \varepsilon e^{i\theta}$.

(iii) Obtain an inequality by studyng the modulus of both sides of your result in (ii).

(iv) Now subtract $|f(z_0)|$ from your inequality and insert the triviality $1 = \frac{1}{2\pi}\int_0^{2\pi} d\theta$ judiciously.

(v) Employ continuity of $f(z)$ and the assumption that $|f(z_0)|$ was a maximum to derive an absurdity.

Question 3 Use Cauchy's integral formula, contour integration and the identity $2\cos(\theta) = z + 1/z$ for $z = \exp(i\theta)$ to show

$$\int_0^{2\pi} \frac{d\theta}{13 - 12\cos\theta} = \frac{2\pi}{5}\,.$$