185A Homework 9

Ungraded: use these problems to practice for the final exam.

Question 1. Use uniform convergence of $1/(1-z) = \sum_{n=0}^{\infty} z^n$ on $|z| \leq R < 1$ to derive power series expansions for $\log(1-z)$ and $1/(1-z)^2$.

Question 2. Find Laurent series for the following functions in the regions indicated

(i) $$f(z) = \frac{z}{(z-1)(z-3)} \qquad \text{for } 0 < |z-1| < 2$$

(ii) $$f(z) = \frac{16}{z^2(z-4)} \qquad \text{for } 0 < |z| < 4 \text{ and } |z| > 4$$

Question 3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converge for $|z| < R$. If $0 < r < R$, show that $f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$, where $z = re^{i\theta}$ and

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

Also show

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

The second equation is known as **Parseval's theorem**.
Hint: Expand $f\overline{f}$ in a series and integrate term by term.

Question 4. Evaluate

$$\oint_\gamma \frac{z^2 + e^z}{z(z-3)} dz$$

where γ is a unit circle about (i) $z = 0$, (ii) $z = 3$.

Question 5. Find and classify the singularities of each of the following functions:

(i) $\dfrac{z^3+1}{z^2(z+1)}$

(ii) $z^3 e^{1/z}$

(iii) $\dfrac{\cos z}{z^2+1}$

(iv) $\dfrac{1}{e^z - 1}$

Question 6. Summarize the key results and theorems, briefly indicating how they were proven.

Question 7. Calculate the Cauchy Riemann relations in polar coordinates $z = r\exp(i\theta)$.

Question 8. What is i^i. Discuss!

Question 9. Develop an asymptotic expansion of $I(t) = \int_0^\infty dx \frac{\exp(-x)}{1+xt}$ valid when t is small.

Question 10. Compute $I = \int_0^\infty \frac{dx}{(1+x^2)^2}$ using contour integration.

Question 11. Compare and contrast differentiability (over $\mathbb{R}$) and analyticity.

Question 12. Develop a power series expansion for $f(z) = \frac{1}{1-z}$ about $z_0 \neq 1$. What is its radius of convergence?

Question 13. Suppose $f(z)$ is analytic at z_0. Give an integral formula for the coefficients of its power series about this point. (Indicate how one derives this result.)

Question 14. Write a power series for $\exp(z)$ about $z = 0$. Deduce the following integrals

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos(\theta)}\cos(\sin\theta - n\theta)d\theta = \frac{1}{n!}$$
$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos(\theta)}\sin(\sin\theta - n\theta)d\theta = 0$$

Question 15. Compute

$$I = \frac{1}{2\pi i}\oint_{\Gamma_z} \frac{dw}{w(w-z)^2}$$

by expanding $1/w$ in a power series about $w = z$.

Question 15. Compute the Laurent series for

$$f(z) = \frac{4}{z^2 - 8z + 12}$$

about $z = 3$ valid in an annulus $1 < |z-3| < 3$ using the integral formula for the coefficients a_n.

Question 16. Define the residue of a function $f(z)$ at $z = z_0$. Indicate why residues are useful. There are some handy rules for computing residues:

- If f has a simple pole at z_0 then

$$\mathrm{res}_{z_0} f = \lim_{z\to z_0} f(z)$$

- If $f = P/Q$ where the functions P and Q are analytic at z_0 and $P(z_0) \neq 0 = Q(z_0)$ where z_0 is a simple zero, then
$$\text{res}_{z_0} f = P(z_0)/Q'(z_0)$$
- If f has a pole order m at z_0 then
$$\text{res}_{z_0} f = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^m}{dz^m} \Big[(z - z_0)^m f(z) \Big]$$
- If $f = g'/g$, $g(z_0) = 0$ and g is analytic at z_0 then
$$\text{res}_{z_0} f = \text{order of the zero of } g$$

Derive each of these rules.

Question 17. Let
$$f(z) = \frac{1}{z^2(z-1)}$$
Compute its residue at $z = 0$ and $z = 1$.

Question 18. Let
$$f(z) = \cot z$$
Compute its residue at $z = 0$.

Question 19. Compute the Fourier transform of $1/(1 + x^2)$. Hint: View the Fourier integral as a contour integral and introduce a large semicircle about $z = 0$ to “close the contour”.