Homework 9 Partial Solutions

Question 1. Use uniform convergence of $1/(1-z) = \sum_{n=0}^{\infty} z^n$ on $|z| \le R < 1$ to derive power series expansions for $\log(1-z)$ and $1/(1-z)^2$.

Solution: Since $1/(1-z) = \sum_{n=0}^{\infty} z^n$ converges uniformly on the disc of radius R, we can take the derivative on either side and they will match. Equating the two sides gives:

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} nz^{n-1}$$

We may also take the integral componentwise, so we have

$$-\log(1-z) = \int \frac{dz}{1-z} = \int \sum_{n=0}^{\infty} z^n dz = \sum_{n=0}^{\infty} \int z^n dz = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Multiplying by (-1) gives the desired power series.

Question 2. Find Laurent series for the following functions in the regions indicated:

$$\text{(i)} \quad f(z) = \frac{z}{(z-1)(z-3)} \qquad \text{for } 0 < |z-1| < 2$$

Solution: (part (i) only) Since we want to expand about the point $z = 1$, we want to get f into a form $\sum_{j=-k}^{\infty} a_j (z-1)^j$. Use partial fractions to rewrite f, and get it into such a form:

$$\begin{aligned} f(z) = \frac{z}{(z-1)(z-3)} &= \frac{-1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{z-3} \\ &= \frac{-1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{z-1+1-3} \\ &= \frac{-1}{2}\frac{1}{z-1} - \frac{3}{2}\frac{1}{2-(z-1)} \\ &= \frac{-1}{2}\frac{1}{z-1} - \frac{3}{4}\frac{1}{1-\frac{z-1}{2}} \\ &= \frac{-1}{2}\frac{1}{z-1} - \frac{3}{4}\sum_{j=0}^{\infty}\frac{(z-1)^j}{2^j} \end{aligned}$$

Where we used the identity: $\frac{1}{1-w} = 1 + w + w^2 + \cdots$ in the last line. We can therefore write f as a Laurent series:

$$f(z) = \frac{-1}{2}\frac{1}{z-1} - \frac{3}{4} - \frac{3}{8}(z-1) - \frac{3}{16}(z-1)^2 - \cdots$$

Question 4. Evaluate

$$\int_\gamma \frac{z^2 + e^z}{z(z-3)} dz$$

where γ is the unit circle.

Solution: We could evaluate the integral directly, by parameterizing γ and performing the real integral that results (although you may get something nasty.) Another way is to use Cauchy's Integral Formula: Let $f(z) = \frac{z^2+e^z}{z-3}$. Then f is analytic away from $z = 3$, so by Cauchy's Integral Formula (since the unit circle does not contain $z = 3$!), we have that $f(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z} dz$ which is exactly the integral we want to evaluate. So all we have to do is compute $f(0) = \frac{0^2+e^0}{0-3} = \frac{-1}{3}$. Then the integral above equals $\frac{-2\pi i}{3}$.

Question 5. Find and classify the singularities of each of the following functions:

(i) $\frac{z^3+1}{z^2(z+1)}$

(ii) $z^3 e^{1/z}$

(iii) $\frac{\cos z}{z^2+1}$

(iv) $\frac{1}{e^z-1}$

Solution: Function (i) has a double pole at 0 and a simple pole at -1.

Function (ii) has an essential singularity at 0 because of the $e^{1/z}$ term. Why? What's the difference between removable singularities and poles and essential singularities? Simply put, removable singularities are singularities z_0 such that f is bounded near z_0. Poles are singularities such that $|f(z)| \to \infty$ as $z \to z_0$. And essential singularities are everything else. If we look at the function $f(z) = e^{1/z}$, we can see it is not defined at $z = 0$ (and only there.) If we approach 0 along the positive real axis, then $f(z) \to \infty$. If we approach along the negative real axis, $f(z) \to 0$. Since it is neither bounded nor goes to infinity along every approach, f must have an essential singularity at 0. (Note: if z approaches 0 along the imaginary axis, $f(z)$ will oscillate but be bounded.)

Function (iii) has simple poles at i and $-i$, as seen by factoring $(z^2 + 1) = (z + i)(z - i)$.

Function (iv) has a singularity at $z = 0$. Since e^z is holomorphic, it doesn't matter how we approach 0, we'll always get $|f(z)| \to \infty$ as $z \to 0$. Therefore, f has a pole at 0 (in fact a simple pole.

Question 7. Calculate the Cauchy-Riemann relations in polar coordinates $z = r\exp(i\theta)$.

Solution: Observe that

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta r,\Delta\theta\to 0}\frac{f(r+\Delta r,\theta+\Delta\theta)-f(r,\theta)}{(r+\Delta r)\exp(i(\theta+\Delta\theta))-r\exp(i\theta)}.$$

Letting $\Delta\theta = 0$, we have

$$\begin{aligned}\lim_{\Delta r\to 0}\frac{f(r+\Delta r,\theta)-f(r,\theta)}{(r+\Delta r)\exp(i\theta)-r\exp(i\theta)} &= \lim_{\Delta r\to 0}\frac{f(r+\Delta r,\theta)-f(r,\theta)}{\Delta r}\exp(-i\theta)\\ &= \exp(-i\theta)\frac{\partial f}{\partial r}.\end{aligned}$$

Letting $\Delta r = 0$, we have

$$\lim_{\Delta\theta\to 0}\frac{f(r,\theta+\Delta\theta)-f(r,\theta)}{r\exp(i(\theta+\Delta\theta))-r\exp(i\theta)} = \lim_{\Delta\theta\to 0}\frac{f(r,\theta+\Delta\theta)-f(r,\theta)}{\exp(i\Delta\theta)-1}\frac{1}{r\exp(i\theta)}:$$

Noting that $r\exp(i\theta) = z$ and

$$\lim_{\Delta\theta\to 0}\frac{\exp(i\Delta\theta)-1}{\Delta\theta} = i,$$

we conclude that the above expression is equal to

$$\frac{1}{iz}\lim_{\Delta\theta\to 0}\frac{f(r,\theta+\Delta\theta)-f(r,\theta)}{\Delta\theta} = \frac{1}{iz}\frac{\partial f}{\partial\theta}$$

The real part of $\frac{df}{dz}$ can be written as either $\exp(-i\theta)\partial_r u$ or $\frac{1}{z}\partial_\theta v$. Multiplying both of these expressions by $\exp(i\theta)$ yields $\partial_r u = \frac{1}{r}\partial_\theta v$.

Similarly, the imaginary part of $\frac{df}{dz}$ can be written as $\exp(-i\theta)\partial_r v$ or $\frac{-1}{z}\partial_\theta u$. Multiplying by $-\exp(i\theta)$ yields $-\partial_r v = \frac{1}{r}\partial_\theta u$.

Question 8. What is i^i? Discuss!

Solution: For $\alpha, z \in \mathbb{C}$, z^α is defined as $z^\alpha = e^{\alpha\log(z)}$ (Note: the log in this definition is the multiple-valued log, without any branch or branch cut chosen.) Since log appears in the definition, that's a good hint the function is multiple-valued and so we will need to choose a branch and branch cut. (Bonus question: How many branches will this function have for $\alpha = \frac{1}{2}$? $\frac{1}{3}$? $\frac{2}{5}$? $\frac{1}{\pi}$?) How do we go about doing this? An example might help: Let $\alpha = \frac{1}{3}$. We first need to find the branch points of the function, which can only occur when the function is 0 or undefined. In our case, we only need check the point $z = 0$, where the function is undefined (since we would need to calculate $\log 0$.) A point is a branch point if, when you travel continuously in a small circle about the point the function takes on different values depending on how many times you've circled. For instance, for the function $f(z) = z^{\frac{1}{3}}$, we can circle 0 and calculate function values:

Starting at $z = \epsilon e^{i0}$, we calculate $f(z) = e^{\frac{1}{3}\log(\epsilon e^{i0})} = e^{\frac{1}{3}\log|\epsilon|}e^{\frac{1}{3}i0} = e^{\frac{1}{3}\log|\epsilon|}$. If we then run continuously around the circle $\epsilon e^{i\theta}$, when we come back to the point ϵ we're actually at $\epsilon e^{i2\pi}$. Then $f(z) = e^{\frac{1}{3}\log(\epsilon e^{i2\pi})} = e^{\frac{1}{3}\log|\epsilon|}e^{\frac{1}{3}i2\pi}$ which is *not* equal to $e^{\frac{1}{3}\log|\epsilon|}$. Just for fun, let's run around the circle a few more times. At $z = \epsilon e^{i4\pi}$, $f(z) = e^{\frac{1}{3}\log|\epsilon|}e^{\frac{1}{3}i4\pi}$, and at $z = \epsilon e^{i6\pi}$, $f(z) = e^{\frac{1}{3}\log|\epsilon|}e^{\frac{1}{3}i6\pi} = e^{\frac{1}{3}\log|\epsilon|}$. So we're back to where we started! What does this show us? First of all, it shows that 0 is in fact a branch point of the function. It also shows that there are three branches – the one where $f(\epsilon) = e^{\frac{1}{3}\log|\epsilon|}$, the one where $f(\epsilon) = e^{\frac{1}{3}\log|\epsilon|}e^{\frac{1}{3}i2\pi}$, and the one where $f(\epsilon) = e^{\frac{1}{3}\log|\epsilon|}e^{\frac{1}{3}i4\pi}$. But we want our functions to be single-valued (otherwise we have to define them on these Riemann surfaces that have multiple branches, which is cool, but hard to picture.) How is that accomplished? By insisting that we don't make a circle around 0. That's the reason for branch cuts, to keep you from circling continuously around branch points, which is where you have trouble. In this case, we could make any cut emanating from 0 and going out to infinity. Once this is done, we have to pick in which branch we're going to live. That is, making the branch cut ensures we can't go between branches, but we still have to decide which of the three branches will define our function. Picking a branch is the same as picking a value for $f(\epsilon)$ (or equivalently, picking a value for $f(z)$ for any z not on our chosen branch cut.) That done, we finally have a well-defined function.

This same kind of analysis can be applied to z^α for any $\alpha \in \mathbb{C}$. If you pick $\alpha \in \mathbb{N}$, you should find that 0 is not a branch point, but in fact the function only takes on one value even after circling 0. Let's study a complex value, say $\alpha = i$. Same as before, we can calculate z^i where we loop around 0. Let's dispense with the ϵ, a circle of radius 1 will do just fine here. So for $z = e^{i0}$, we get $z^i = e^{i\log(e^{i0})} = e^{ii0} = 1$. For $z = e^{i2\pi}$, we end up with $z^i = e^{i\log(e^{i2\pi})} = e^{-2\pi}$. If we continue on, it's not too hard to see that for $z = e^{i2n\pi}$, $z^i = e^{-2n\pi}$. So 0 is a branch point and there are an infinite number of branches of the function z^i. So to define i^i, we need to pick a branch cut not containing i, and then pick a branch of the function. An obvious choice is to pick $(-\infty, 0]$ as the branch cut, and pick the branch where $1^i = 1$. This would give us that $i^i = (e^{i\pi/2})^i = e^{-\pi/2}$. However, we could just as easily pick the branch cut to be the ray $x = y$ from 0 to infinity, and pick the branch such that $1^i = (e^{i4\pi})^i = e^{-4\pi}$. Then to move continuously within this branch, without crossing the branch cut, we have to move clockwise along the unit circle, from $e^{i4\pi}$ down to $e^{i5\pi/2}$, so then we get that $i^i = e^{-5\pi/2}$. Either answer is valid – it's just necessary to define your function accurately.

Question 10. Compute $I = \int_0^\infty \frac{dx}{(1+x^2)^2}$ using contour integration.

Solution: Let's look at the complex function $f(z) = \frac{1}{(1+z^2)^2}$. Define a contour γ as the path going from $-R$ to R along the x-axis, and then going back to $-R$ by tracing a half-circle above the x-axis. A parameterization for the two parts of this path would be:

$$\gamma_1(t) = t, t = -R..R$$

$$\gamma_2(t) = Re^{it}, t = 0..\pi$$

Since f has a singularity at $z = i$, which lies inside γ, we have that $\int_\gamma f(z)dz = 2\pi i \text{res}_i f(z)$ by the residue formula. Now note that if we let R go to infinity, $\int_{\gamma_1} f(z)dz = 2I$ (using the symmetry of f for $x < 0$ and $x > 0$.) Next we look at the other part of the contour, and see that since:

$$\int_{\gamma_2} f(z)dz = \int_0^\pi \frac{iRe^{it}}{(1 + R^2e^{i2t})^2} \leq \frac{\pi|R|}{|R|^4},$$

the integral over the half-circle portion goes to 0 as R goes to infinity. So what we have, then, is that $I = (\pi i)\text{res}_i f(z)$. And finally, since f has a pole of order 2 at i (since $\frac{1}{(1+z^2)^2} = \frac{1}{((z+i)(z-i))^2}$) we can calculate $\text{res}_i f = \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z)$ (this is a theorem in most books), which evaluates to: $\lim_{z\to i} \frac{d}{dz}\frac{1}{(z+i)^3} = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{2}{8i} = \frac{1}{4i}$. Plugging this into our formula for I gives us that $I = \frac{\pi}{4}$.

Question 13. Suppose $f(z)$ is analytic at z_0. Give an integral formula for the coefficients of its power series about this point. (Indicate how one derives this result.)

Solution: Since f is analytic at z_0, it is analytic in an open neighborhood N of z_0 (by definition.) Let D be an open disc containing z_0 whose closure is contained in N. Then we have Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw$$

for all z in D, where C is the boundary of D. First we rewrite

$$\frac{1}{w-z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{1}{w-z_0}\frac{1}{1 - \frac{z-z_0}{w-z_0}}$$

where $w \in C$, and z lies inside D. Since z is inside the circle and w lies on the circle, there is some $0 < r < 1$ such that $|\frac{z-z_0}{w-z_0}| < r$, so we can use the geometric series expansion to write:

$$\frac{1}{1 - \frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$$

which converges uniformly for all $w \in C$. Plug this into Cauchy's Integral Formula for $f(z)$:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)}dw \quad = \quad \frac{1}{2\pi i}\int_C f(w)\frac{1}{w-z_0}\frac{1}{1 - \frac{z-z_0}{w-z_0}}$$

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_C f(w)\frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_C f(w)\frac{1}{w-z_0}\left(\frac{z-z_0}{w-z_0}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n
\end{aligned}$$

where we can exchange the sum and the integral because of uniform convergence. We therefore have a power series expansion of f about z_0 with an integral formula for the coefficients. Note that the "extended" version of Cauchy's Integral Theorem, $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw$ gives us another formula for the coefficients as derivatives that parallels that for real power series:

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

where the coefficients are derivatives of f.

Question 15. Compute

$$I = \frac{1}{2\pi i}\int_{\Gamma_z}\frac{dw}{w(w-z)^2}$$

by expanding $1/w$ in a power series about $w = z$.

Solution: We don't actually have to calculate the power series expansion of $1/w$, but just recognize I as a certain derivative. Cauchy's theorem gives us a formula for all derivatives of f in terms of integrals:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw$$

where C is a circle containing w. From this we see that if $f(w) = 1/w$, $I = f'(z)$. But we know how to calculate $f'(w)$, it's simply $-1/w^2$. Therefore, $I = -1/z^2$.

Question 17. Let

$$f(z) = \frac{1}{z^2(z-1)}$$

Compute its residue at $z = 0$ and $z = 1$.

Solution: Since f has a simple pole at $z = 1$, its residue is simply $\lim_{z\to 1}(z-1)f(z) = 1$. The pole at $z = 0$ is of order 2, so we compute:

$$\text{res}_0 f \;=\; \lim_{z\to 0}\frac{1}{(2-1)!}\left(\frac{d}{dz}\right)^{2-1} z^2 f(z)$$

$$
\begin{aligned}
&= \lim_{z\to 0} \frac{-1}{(z-1)^2} \\
&= -1
\end{aligned}
$$