21D Vector Calculus

Text: Thomas' Calculus

Syllabus: Multiple Integrals (ch. 15)

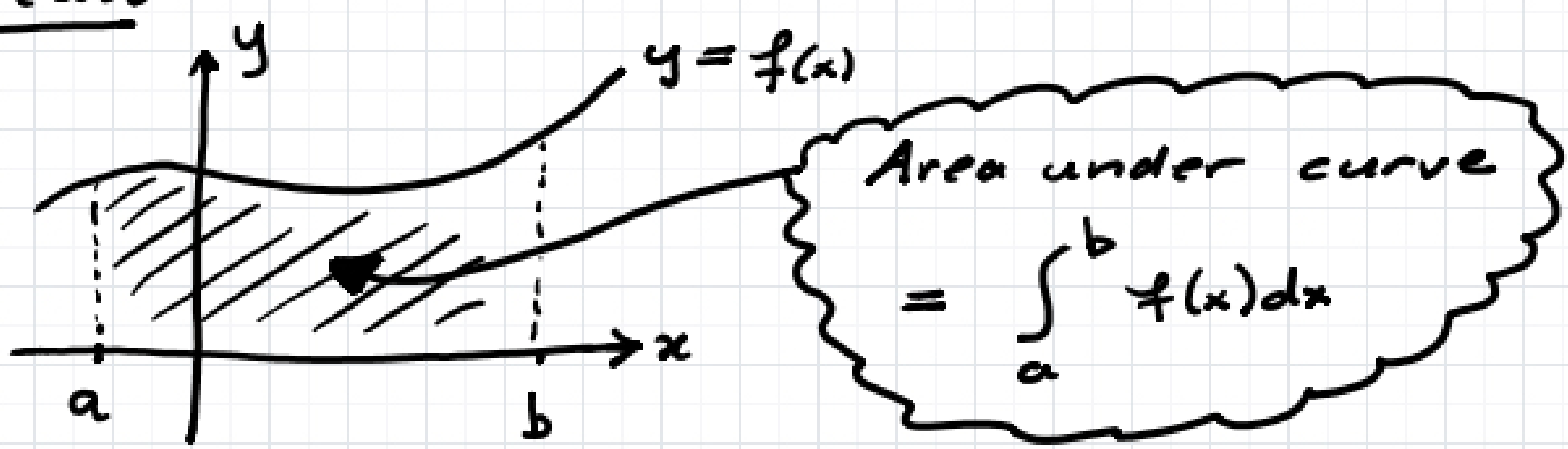
Vector Fields (ch. 16)

Calculus of the world around us.

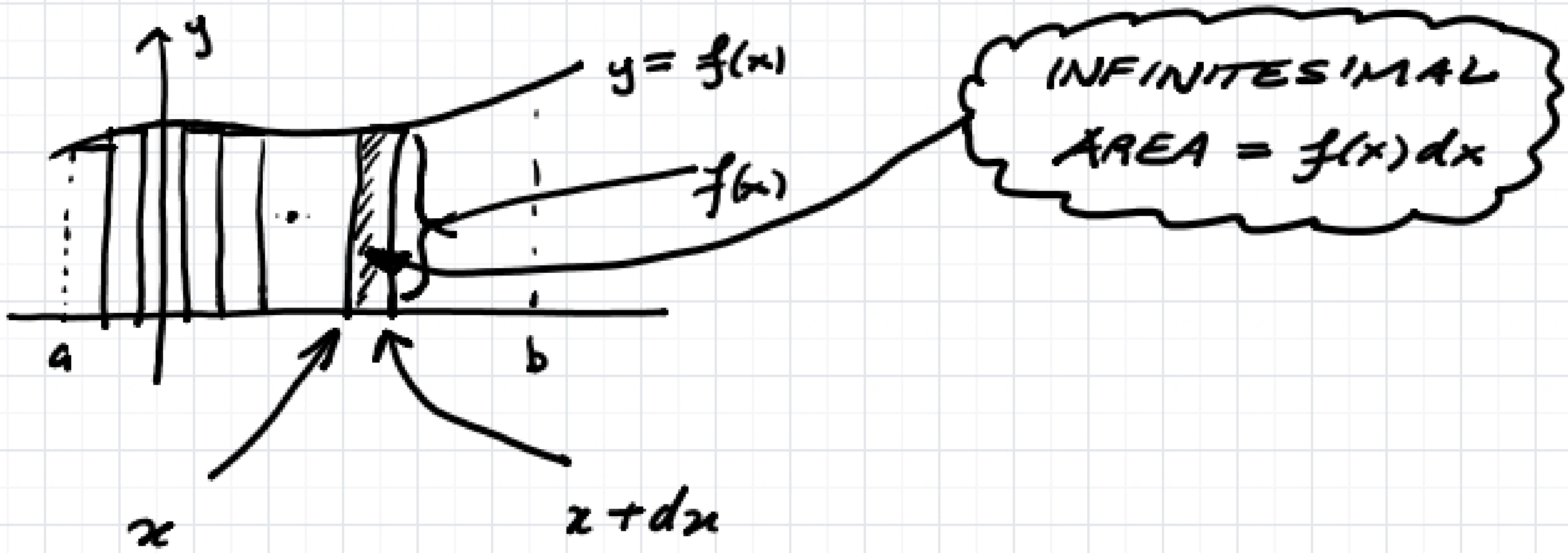
http://www.math.ucdavis.edu/~wally/teaching/21D/21D.html

IMPORTANT DATES ETC...

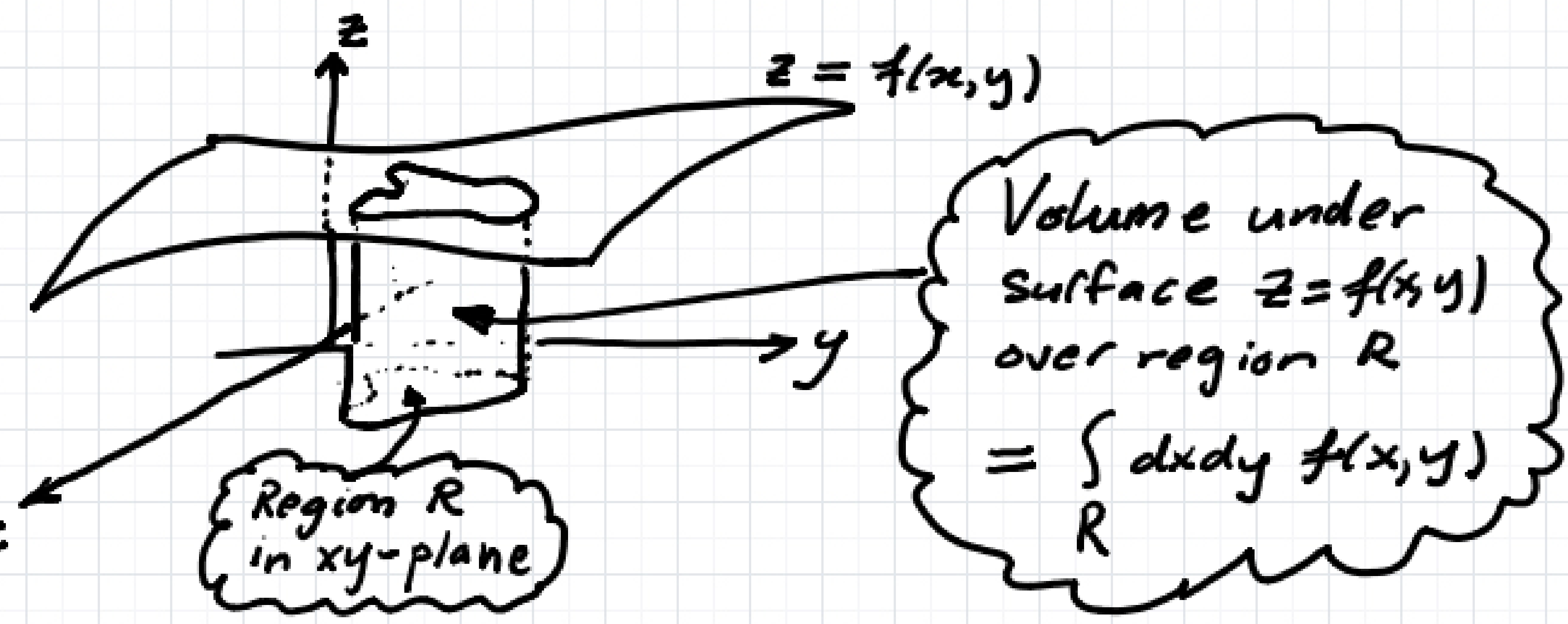
Recall



Riemann sum = limit of sum over rectangles

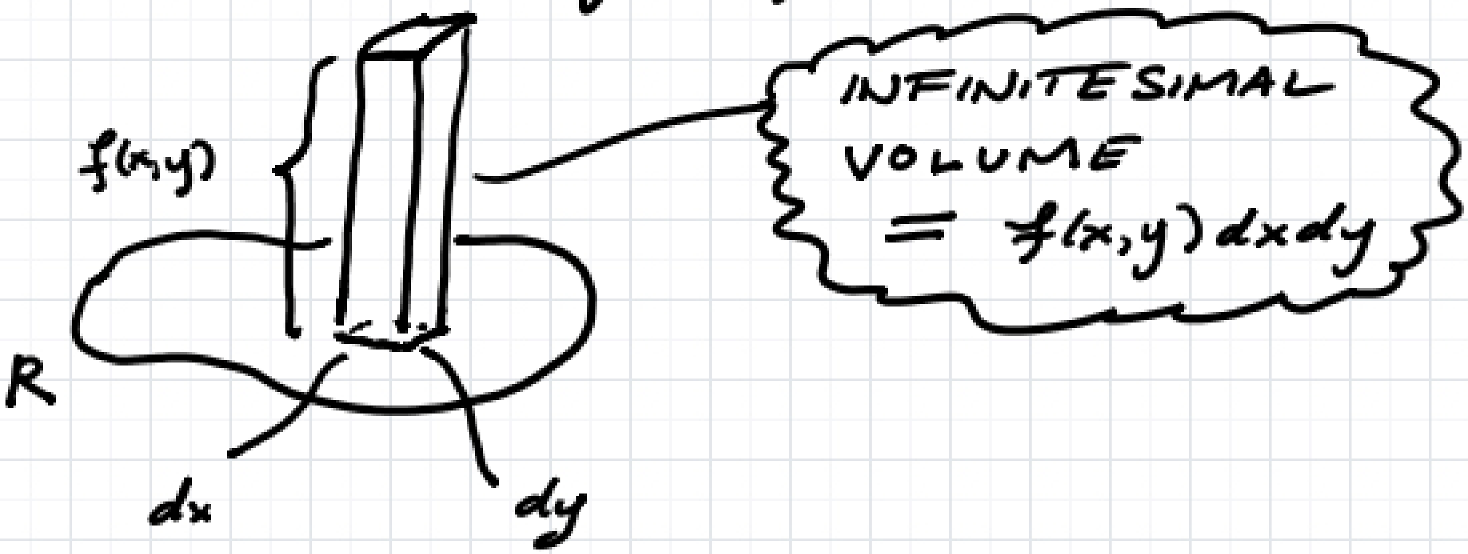


Now consider

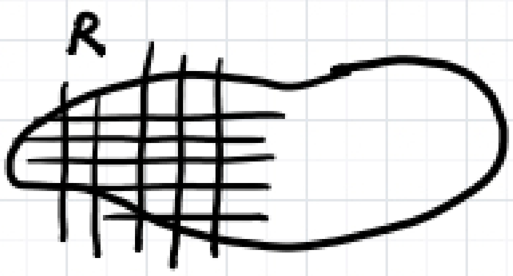


Tricky because although the <u>problem</u> <u>is fixed</u>

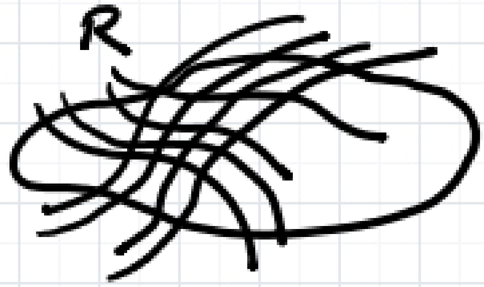
There are many ways to tackle it



<u>Assumes</u> we tile R with infinitesimal rectangles



VERSUS PERHAPS



Really should write

$$\int_R dx\,dy\, f(x,y) = \int_R dA\, f(x,y) \qquad \Big[= \int_R \omega \;,\; \underset{\text{2-form}}{\omega = dx\wedge dy\, f}\Big]$$

Example $R = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$

R is a rectangle

$$f(x,y) = 1$$

$$\int_R dA\, f = \int_R dx\, dy = \int_a^b dx \int_c^d dy = (b-a)(d-c)$$

* When $f(x,y) = 1$, $\int_R dA\, f$ = Area of R

* Usually much harder to determine the limits of (x,y) integrations.

FUBINI'S THEOREM (I)

If $f : R \longrightarrow \mathbb{R}$ is <u>continuous</u>

$$\begin{array}{ccc} \mathrel{\rotatebox[origin=c]{90}{$\in$}} & & \mathrel{\rotatebox[origin=c]{90}{$\in$}} \\ (x,y) & \longmapsto & f(x,y) \end{array}$$

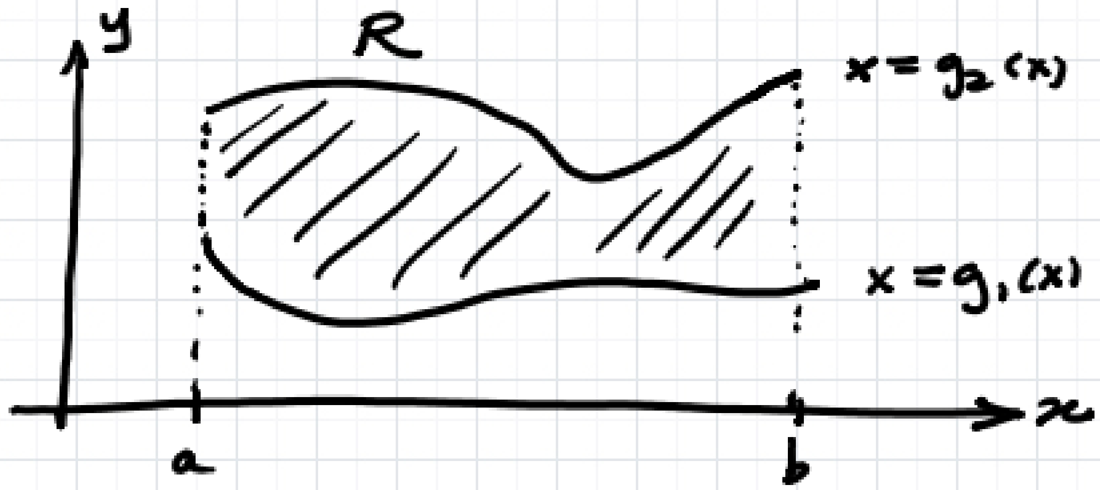
and $R = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$ is a rectangle

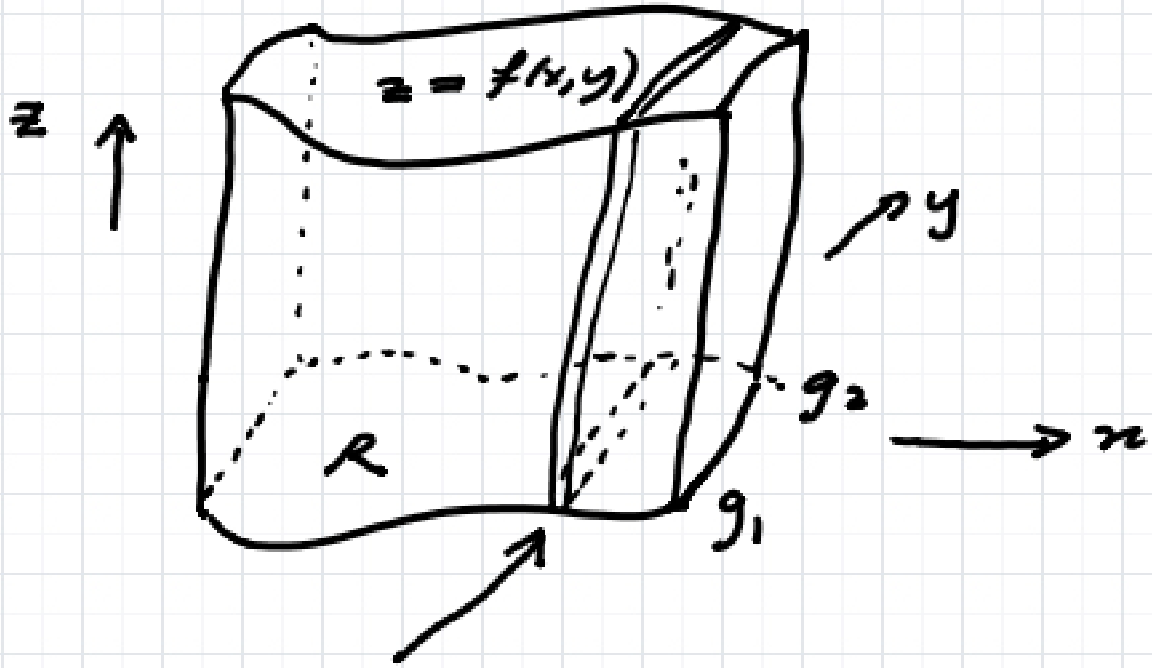
then
$$\int_R dA\, f = \int_a^b dx \int_c^d dy\, f(x,y) = \int_c^d dy \int_a^b dx\, f(x,y)$$

INTEGRAL OVER REGION R = ITERATED INTEGRAL

Example $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

a region bounded by curves





slice thickness dx

Volume of slice at position x $\quad dx \int_{g_1(x)}^{g_2(x)} dy\, f(x,y)$

Now add slices!

FUBINI'S THEOREM

$f : R \longrightarrow \mathbb{R}$ continuous

$R = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(y)\}$

then

$$\boxed{\int_R dA\, f = \int_a^b dx \int_{g_1(x)}^{g_2(x)} dy\ f(x,y)}$$

Adding Slices

* The double integral, or integral over a region R bounded by curves is an iterated integral.

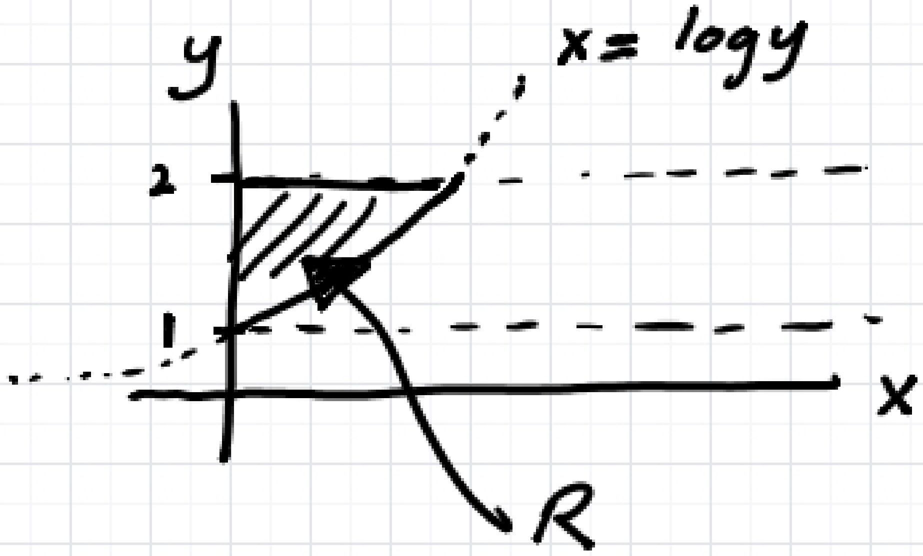
* WARNING

$$\int_R dA\, f = \int_{g_1(x)}^{g_2(x)} dy \int_a^b dx\ f(x,y) \text{ is NONSENSE!}$$

Example $f(x,y) = e^x \log y$

$R = \{1 \le y \le 2,\ 0 \le x \le \log y\}$, find $\int_R dA\, f$

① Sketch R



② Choose how to slice! Here use dy slices

③ Set up iterated integral

$$\int_R dA\, f = \int_1^2 dy \int_0^{\log y} dx\, e^x \log y = \int_1^2 dy \left[\log y\, e^x\right]_0^{\log y}$$

$$= \int_1^2 dy \log y \left[e^{\log y} - 1\right] = \int_1^2 dy \left[y \log y - \log y\right]$$

$$= \frac{1}{4}$$

Presentation like this ain't gonna cut it on the final!!

$$\int_1^2 dy\, y \log y$$

$$= \int_1^2 d\left(\tfrac{1}{2}y^2\right)\log y$$

$$= \left[\left(\tfrac{1}{2}y^2 \log y\right)\right]_1^2 - \int_1^2 \tfrac{1}{2}y^2\, \underbrace{d(\log y)}_{\frac{1}{y}}$$

$$= \tfrac{4}{2}\log 2 - \int_1^2 \tfrac{1}{2}y$$

$$= 2\log 2 - \tfrac{1}{4}(4-1)$$

$$\frac{I}{4}$$

$$\int_1^2 dy \log y = \left(y\log y - y\right)\Big|_1^2 = 2\log 2 - 1$$