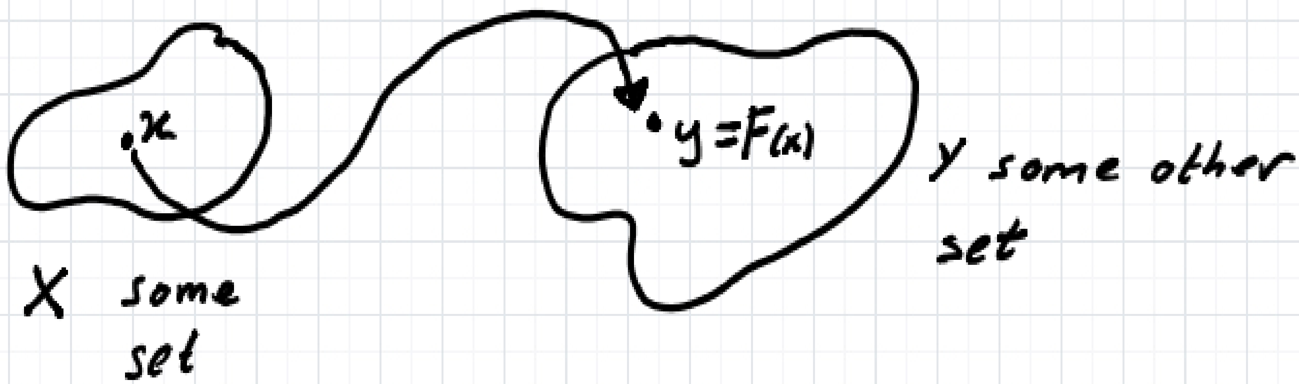
LECTURE #10

– DISCUSS MIDTERM I –

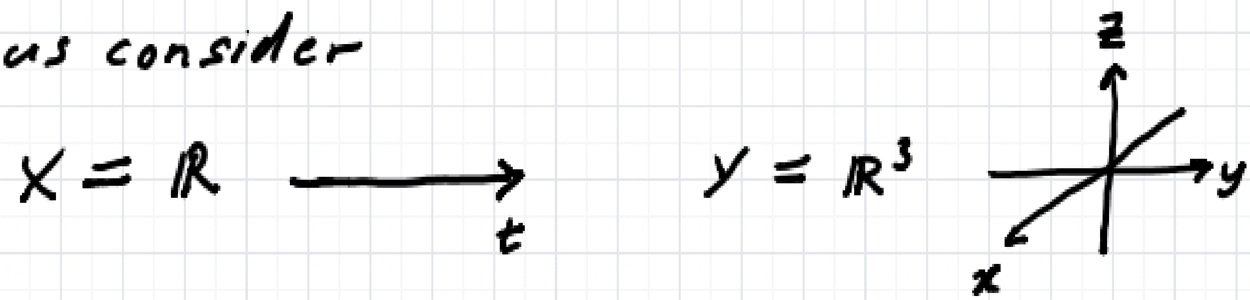
VECTOR FUNCTIONS ch. 13.1

Recall function $F : X \longrightarrow Y$

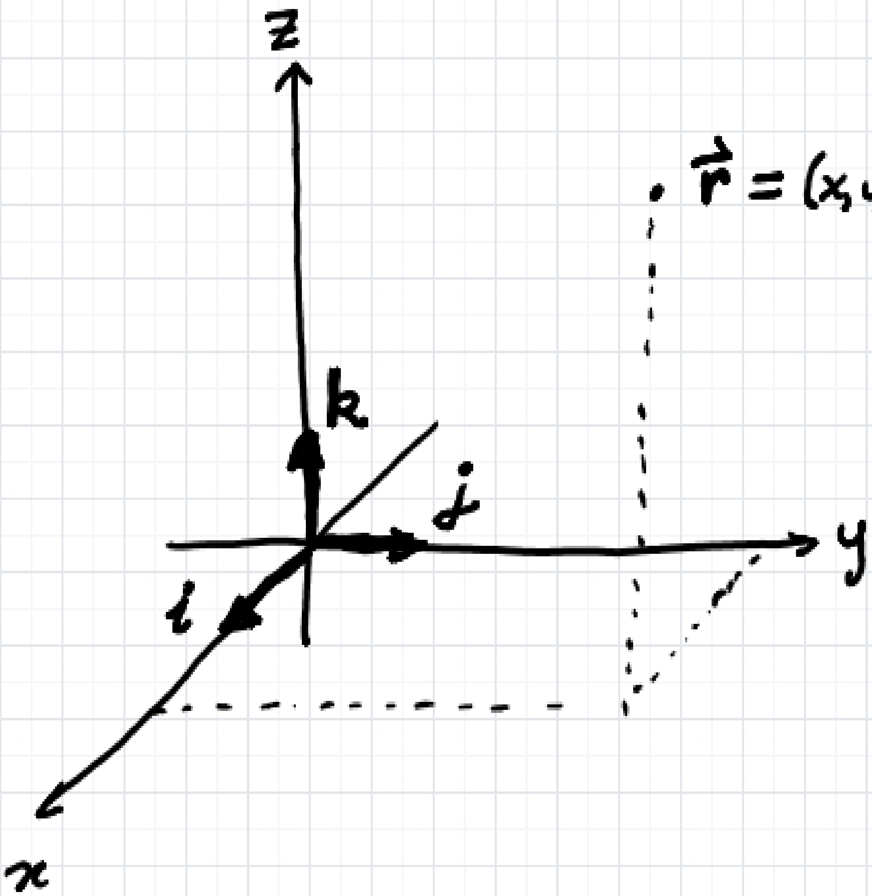


– single output $y = f(x)$ for each input x

Let us consider



Note points in $\mathbb{R}^3$ are vectors



$$\vec{r} = (x, y, z) = xi + yj + zk$$

Visualizing vector functions

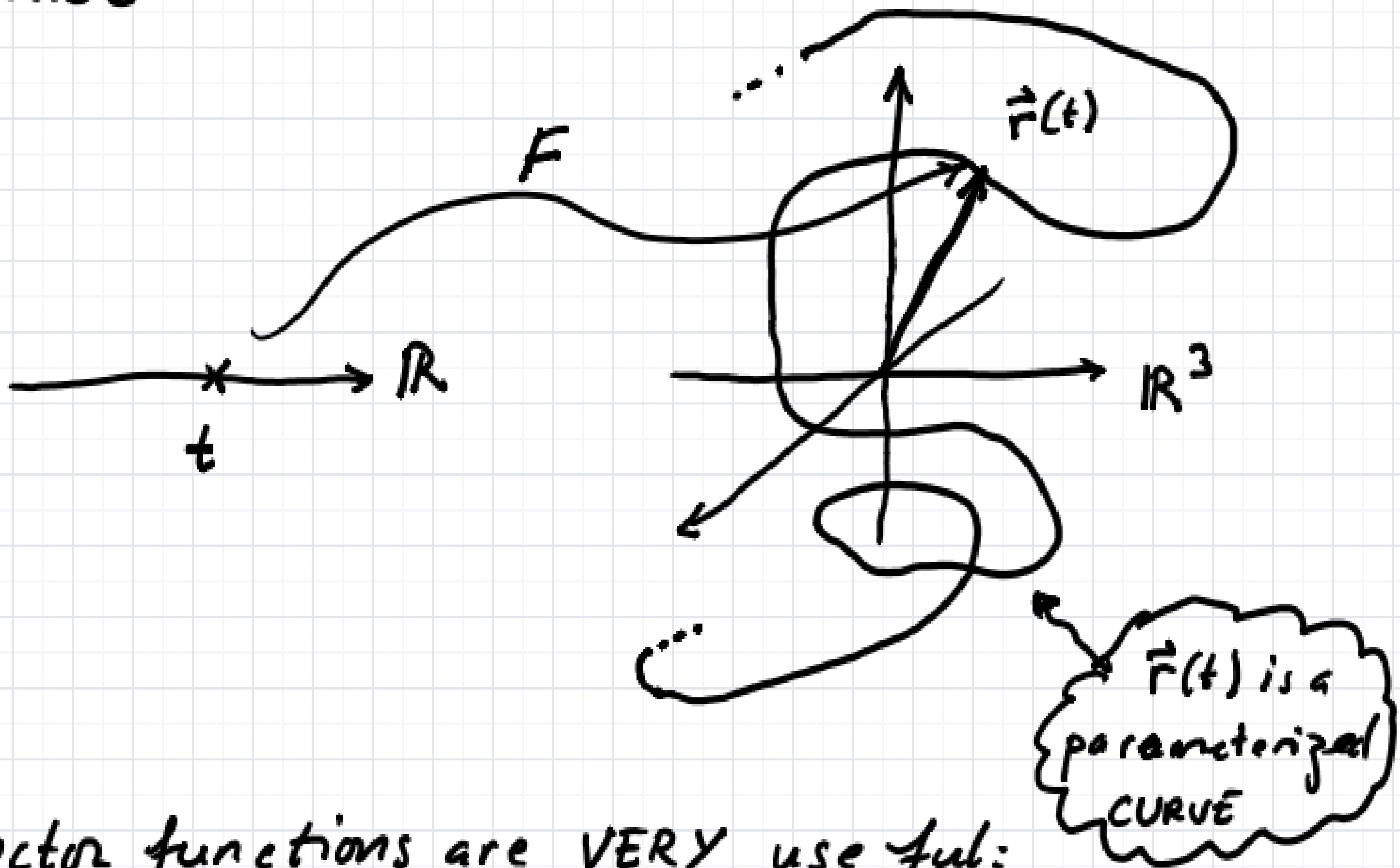
TIME → $\mathbb{R}$ SPACE → $\mathbb{R}^3$

$$F : \mathbb{R} \longrightarrow \mathbb{R}^3$$

$$t \longmapsto F(t) = \vec{r}(t) = f(t)i + g(t)j + h(t)k$$

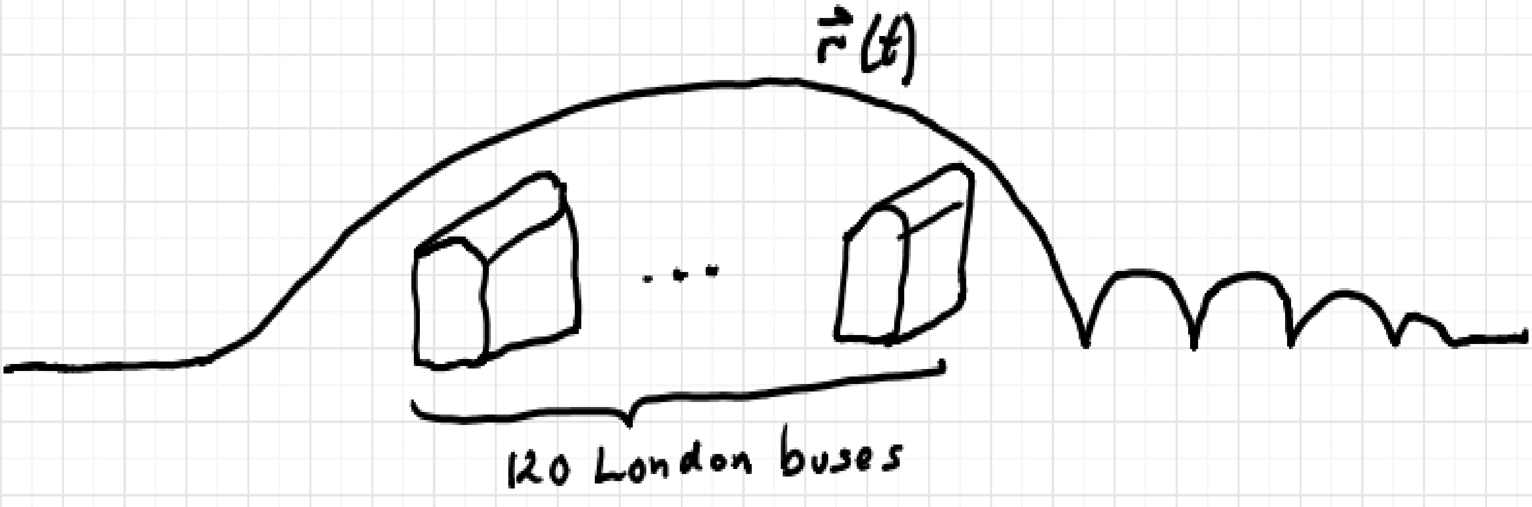
3 ordinary functions

As time t passes, $\vec{r}(t)$ moves through space



Vector functions are VERY useful:

EVIL KNIEVEL



LIMITS, CONTINUITY & DERIVATIVES

The length of a vector $\vec{v} = v_1 i + v_2 j + v_3 k$ is given by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\vec{v}\cdot\vec{v}}$$

Hence we say $\lim\limits_{t \to t_0} \vec{r}(t) = \vec{L}$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that

$$0 < |t - t_0| < \delta \Rightarrow |\vec{r}(t) - \vec{L}| < \varepsilon$$

number

limit of a vector function is a vector

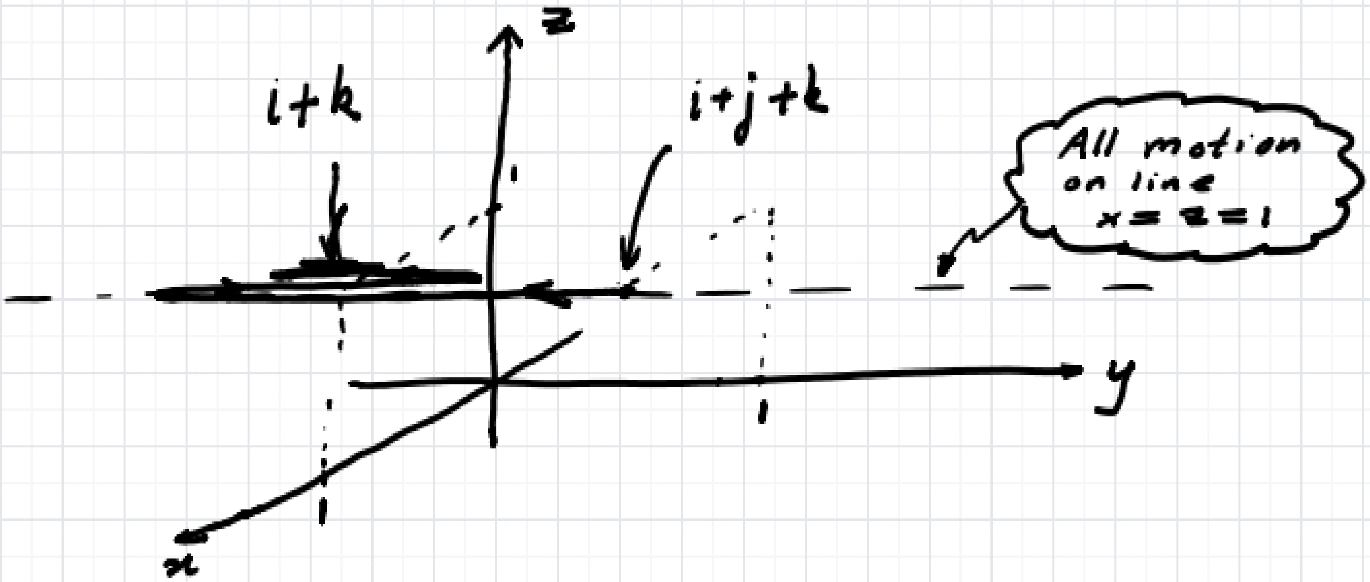
number

We say $\vec{r}(t)$ is <u>continuous</u> at $t = t_0$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

for t_0 in the domain of $\vec{r}(t)$.

Example $\vec{r}(t) = i + \frac{\sin t}{t} j + k$



Find (i) $\lim_{t \to 0} \vec{r}(t)$ (ii) $\lim_{t \to \infty} \vec{r}(t)$

(i) $$\lim_{t \to 0} \left(i + \frac{\sin t}{t} j + k \right) = i + \left(\lim_{t \to 0} \frac{\sin t}{t} \right) j + k$$

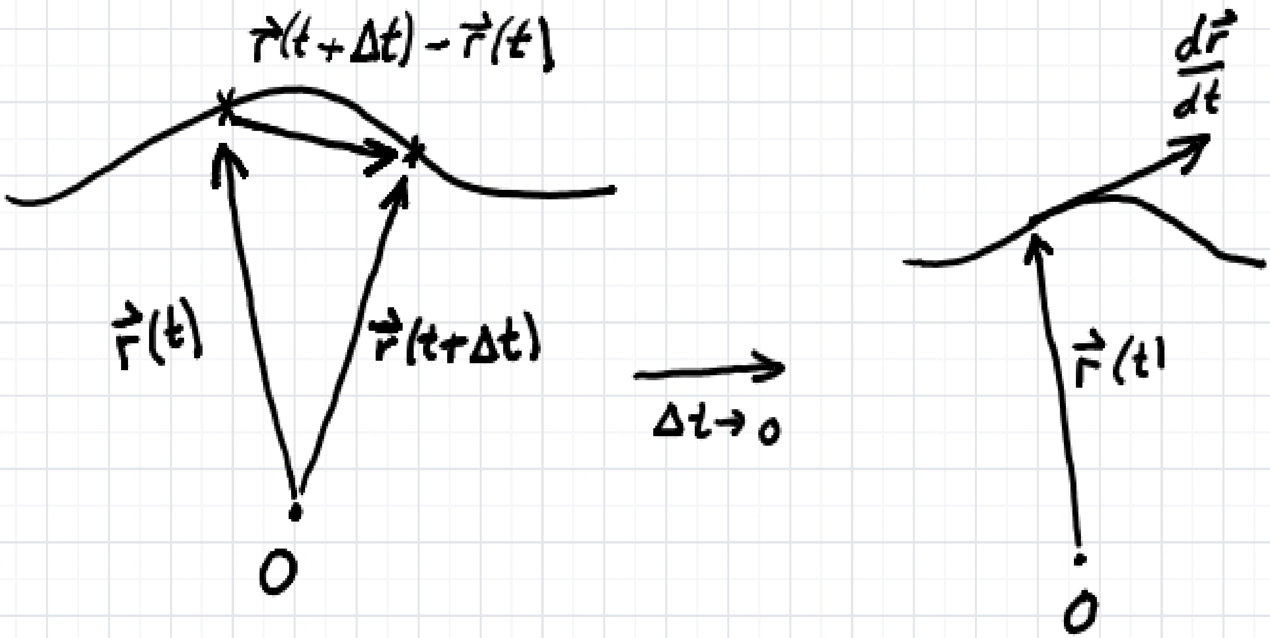
$$= i + j + k$$

(ii) $$\lim_{t \to \infty} \left(i + \frac{\sin t}{t} j + k \right) = i + \left(\lim_{t \to \infty} \frac{\sin t}{t} \right) j + k$$

$$= i + k$$

The derivative of a vector function is defined by the usual limit

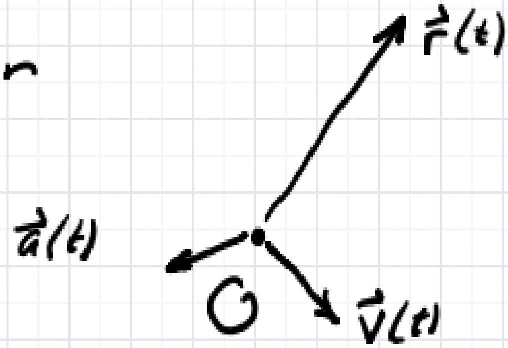
$$\frac{d\vec{r}}{dt} \equiv \vec{r}\,'(t) \equiv \dot{\vec{r}}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$



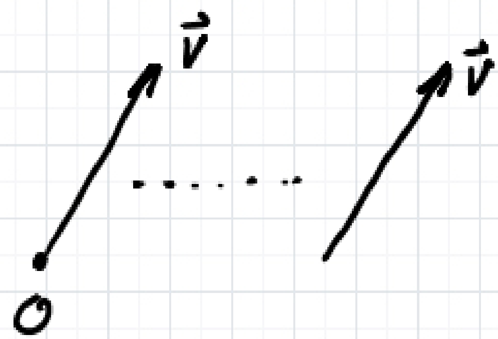
<u>i.e.</u> $\frac{d\vec{r}}{dt}$ is the TANGENT VECTOR

$$\begin{cases} \dot{\vec{r}} = \vec{v} & \text{VELOCITY} \\ |\dot{\vec{r}}| = s & \text{SPEED} \\ \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = \vec{a} & \text{ACCELERATION} \end{cases}$$

Note Should really draw ALL vectors at the origin



But in flat space $\mathbb{R}^3$, we can move vectors around while holding them parallel



(useful for vector addition)

NOTICE This fails on the sphere

INTEGRATION

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{r}_0$$

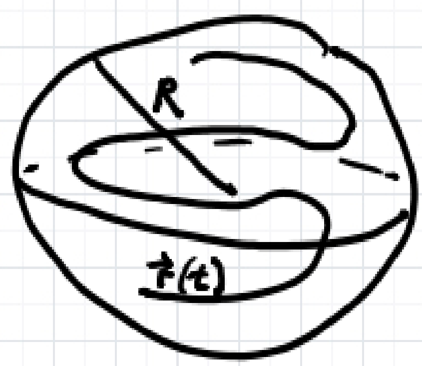
where $\vec{r}_0$ is constant $\dot{\vec{r}}_0 = 0$

and $\vec{R}(t)$ is an antiderivative $\dot{\vec{R}} = \vec{r}(t)$

VECTORS ON THE SPHERE

Suppose $|\vec{r}(t)| = R \quad \forall\ t$.

Then $\vec{r}$ describes motion on a radius R sphere



NB Since $\vec{r}\cdot\vec{r} = R^2$ then

$$0 = \frac{d}{dt}(R^2) = \dot{\vec{r}}\cdot\vec{r} + \vec{r}\cdot\dot{\vec{r}} = 2\vec{r}\cdot\dot{\vec{r}}$$

$\Rightarrow \quad \vec{r}\cdot\dot{\vec{r}} = 0$

i.e. $\dot{\vec{r}}$ is tangent to the sphere