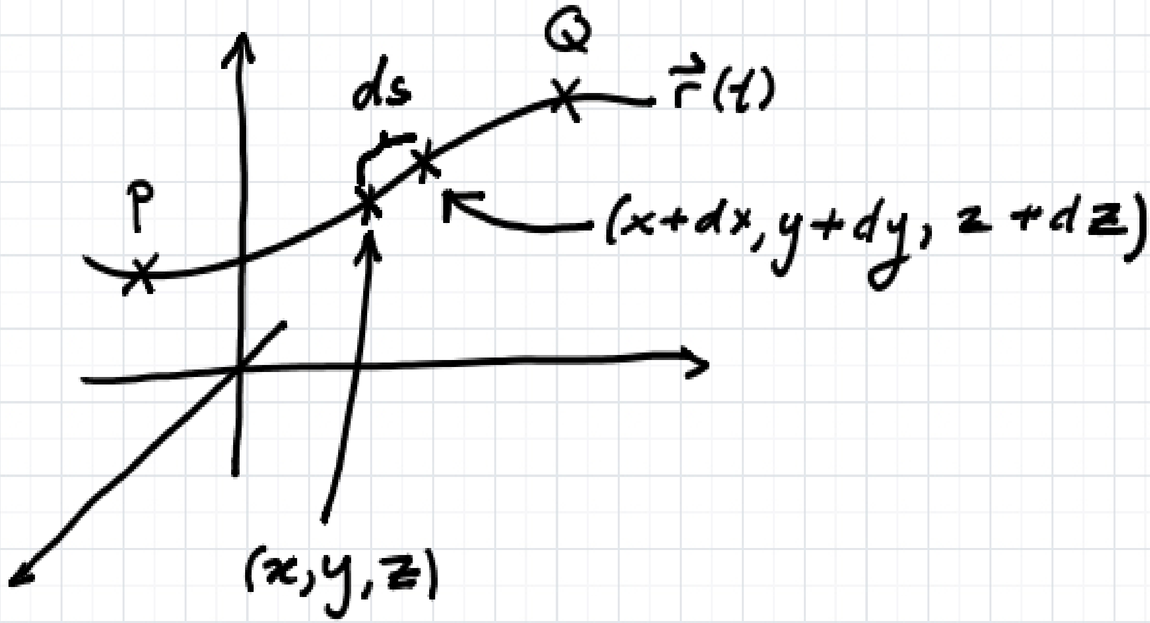
LECTURE #11

ARC LENGTH & TANGENT VECTORS ch. 13.3

Infinitesimal arc length



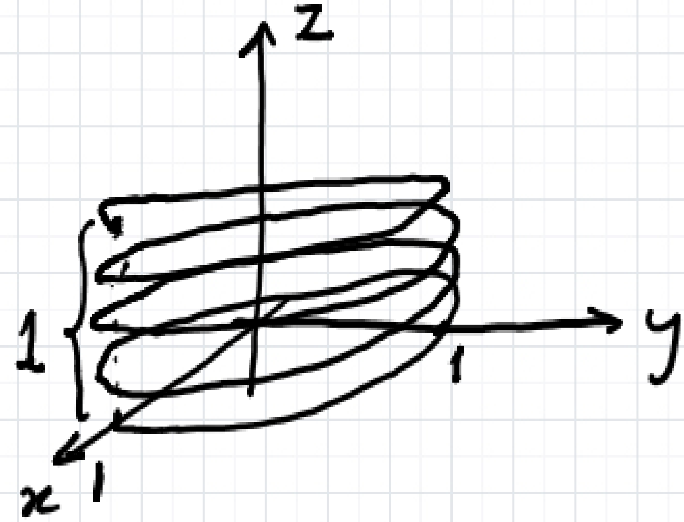
By Pythagoras $ds = \sqrt{dx^2 + dy^2 + dz^2}$

The arc length between points P & Q is

$$L = \int_P^Q ds = \int_P^Q \frac{ds}{dt}\, dt = \int_{t_P}^{t_Q} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_P^Q |\vec{v}|\, dt$$

result cannot depend on choice of parameter t

Example Find the length of the spring shown



1) **Choose** parameterization

$$\vec{r}(t) = \cos t \cdot i + \sin t\, j + \frac{t}{8\pi} k$$

$$0 \leq t \leq 4 \cdot 2\pi$$

4 revolutions .

$$\dot{\vec{r}} = -\sin t\, i + \cos t\, j + \frac{1}{8\pi} k$$

$$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + \frac{1}{64\pi^2}} = \sqrt{1 + \frac{1}{64\pi^2}}$$

$$L = \int_0^{8\pi} \left[1 + \frac{1}{64\pi^2}\right]^{\frac{1}{2}} = 8\pi \left[1 + \frac{1}{64\pi^2}\right]^{\frac{1}{2}}$$

Notice, in principle we could always arrange for the speed

$$s = |\vec{v}| = |\dot{\vec{r}}|$$

to be constant by an appropriate choice of parameterization $\Rightarrow$ $\boxed{L \propto \text{time elapsed}}$

UNIT TANGENT VECTOR

Remember that $\vec{v} = \dot{\vec{r}}$ is the tangent vector. If we only care about the direction of travel then we can compute the unit tangent vector

$$\hat{v} \equiv \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

<u>Example</u> Compute unit tangent vectors for motion along lines of equal (i) r, θ (ii) θ, φ (iii) $r\ \varphi$ in a spherical coordinate system

<u>Solution</u>

$$\vec{r} = r\cos\theta\sin\varphi\, i + r\sin\theta\sin\varphi\, j + \cos\varphi\, k$$

(i) 

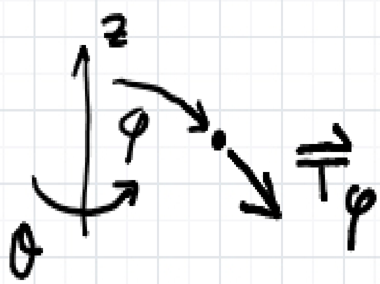
$$\varphi = t$$

$$\vec{r}(t) = r\cos\theta\sin t\, i + r\sin\theta\sin t\, j + r\cos t\, k$$

$$\dot{\vec{r}} = +r\cos t\,(\cos\theta\, i + \sin\theta\, j) - r\sin t\, k$$

$$|\dot{\vec{r}}| = r \quad (\underline{\text{check}})$$

$$\vec{T}_\varphi \underset{\substack{\text{put } t=\varphi \\ \text{again}}}{=} \cos\varphi\cos\theta\, i + \cos\varphi\sin\theta\, j - \sin\varphi\, k$$



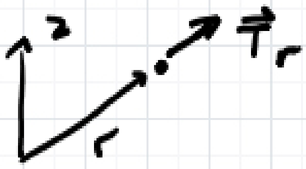
(ii) 

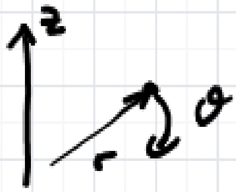
$r = t$

$\dot{\vec{r}} = \sin\varphi\cos\theta\, i + \sin\varphi\sin\theta\, j + \cos\varphi\, k$

$|\dot{\vec{r}}| = 1$

$$\vec{T}_r = \sin\varphi\cos\theta\, i + \sin\varphi\sin\theta\, j + \cos\varphi\, k$$

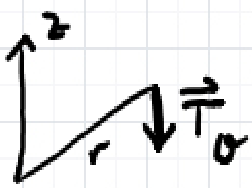


(iii) 

$\theta = t$

$\dot{\vec{r}} = -\sin\varphi\sin\theta\, i + \sin\varphi\cos\theta\, j$

$|\dot{\vec{r}}| = |\sin\varphi| = \sin\varphi \qquad 0 \le \varphi \le \pi$

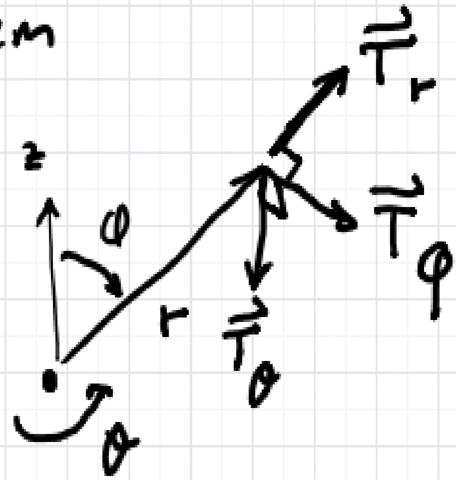


$$\vec{T}_\theta = -\sin\theta\, i + \cos\theta\, j$$

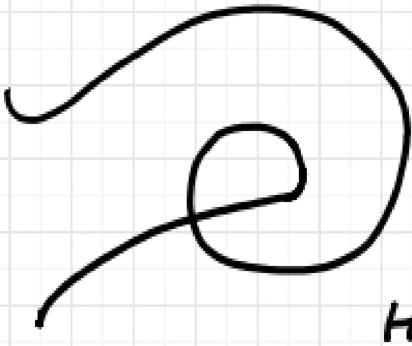
NO VERTICAL COMPONENT

CLAIM $\quad \vec{T}_r \cdot \vec{T}_\theta = \vec{T}_r \cdot \vec{T}_\varphi = \vec{T}_\theta \cdot \vec{T}_\varphi = 0$ orthogonal

coordinate system

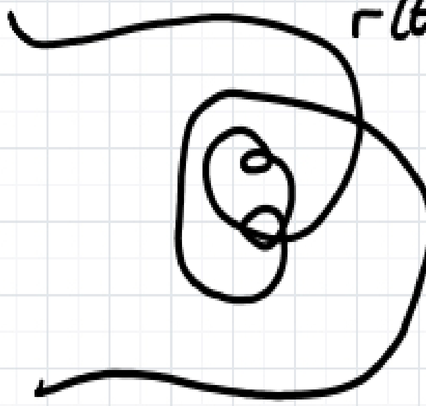


CURVATURE & UNIT NORMAL ch 13.4



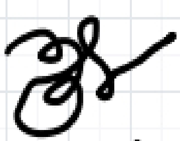
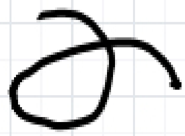
$\vec{r}(t)$

VS



$\vec{r}(t)$

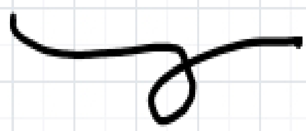
HOW CURVED?

If you traveled at constant speed you could detect the difference between  and  with your eyes shut by examining your acceleration

BUT

SLOWLY

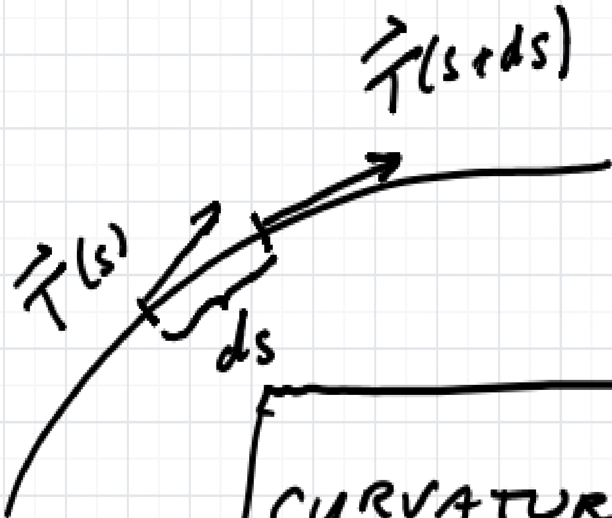
feels nicer than



FAST !!

(Think roller coaster)

Hence consider the rate of change of the <u>unit tangent</u> vector $\vec{T}$ as you move along the path:



s = arc length [NOT SPEED]

$$\boxed{\text{CURVATURE} \quad K = \left|\frac{d\vec{T}}{ds}\right|}$$

But $\dfrac{d\vec{T}}{ds} = \dfrac{d\vec{T}}{dt}\dfrac{dt}{ds} = \dfrac{d\vec{T}}{dt}\dfrac{1}{ds/dt} = \dfrac{\dot{\vec{T}}}{|\vec{v}|}$

$$K = \frac{|\dot{\vec{T}}|}{|\vec{v}|}$$

PARAMETERIZATION INDEPENDENT

<u>Example next time</u>

INTEGRATION

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{r}_0$$

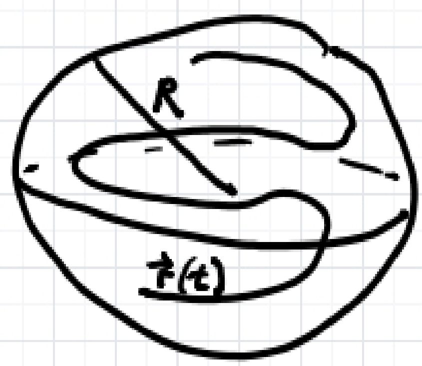
where $\vec{r}_0$ is constant $\dot{\vec{r}}_0 = 0$

and $\vec{R}(t)$ is an antiderivative $\dot{\vec{R}} = \vec{r}(t)$

VECTORS ON THE SPHERE

Suppose $|\vec{r}(t)| = R \quad \forall\ t$

Then $\vec{r}$ describes motion on a radius R sphere



NB Since $\vec{r}\cdot\vec{r} = R^2$ then

$$0 = \frac{d}{dt}(R^2) = \dot{\vec{r}}\cdot\vec{r} + \vec{r}\cdot\dot{\vec{r}} = 2\vec{r}\cdot\dot{\vec{r}}$$

$$\Rightarrow \quad \vec{r}\cdot\dot{\vec{r}} = 0$$

i.e. $\dot{\vec{r}}$ is tangent to the sphere