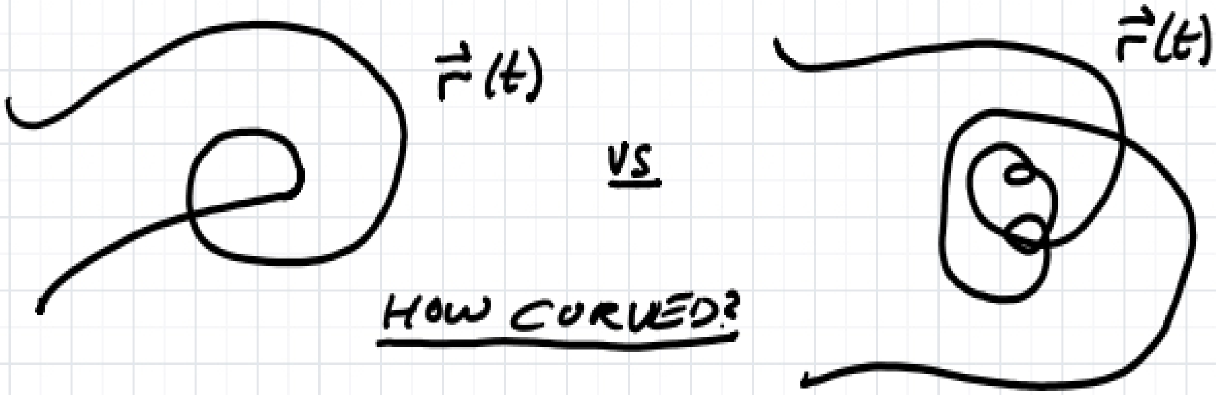
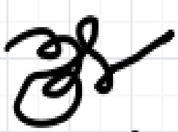
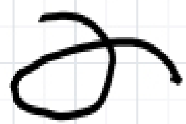
LECTURE #12

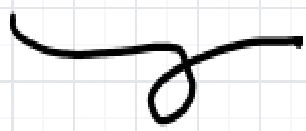
CURVATURE & UNIT NORMAL ch 13.4



If you traveled at <u>constant speed</u> you could detect the difference between  and  with your eyes shut by examining your acceleration

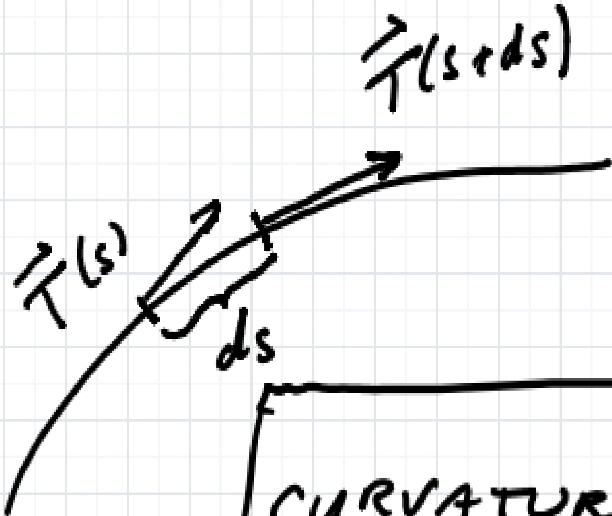
BUT

feels nicer than



(Think roller coaster)

Hence consider the rate of change of the <u>unit tangent</u> vector $\vec{T}$ as you move along the path:



s = arc length [NOT SPEED]

$$\boxed{\text{CURVATURE} \quad K = \left|\frac{d\vec{T}}{ds}\right|}$$

But $$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\frac{dt}{ds} = \frac{d\vec{T}}{dt}\frac{1}{ds/dt} = \frac{\dot{\vec{T}}}{|\vec{v}|}$$

$$K = \frac{|\dot{\vec{T}}|}{|\vec{v}|}$$ PARAMETERIZATION INDEPENDENT

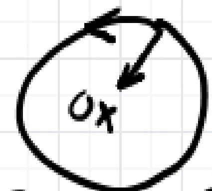
<u>EXAMPLE</u> Circle radius R

$$\vec{r}(t) = R(\cos t\, i + \sin t\, j)$$ in x-y plane (say)

$$\dot{\vec{r}} = R(-\sin t\, i + \sin t\, j) \quad , \quad |\dot{\vec{r}}| = R = |\vec{v}|$$

$$\vec{T} = -\sin t\, i + \sin t\, j$$

$$\dot{\vec{T}} = -(\cos t\, i + \sin t\, j) = -\frac{1}{R}\vec{r}$$



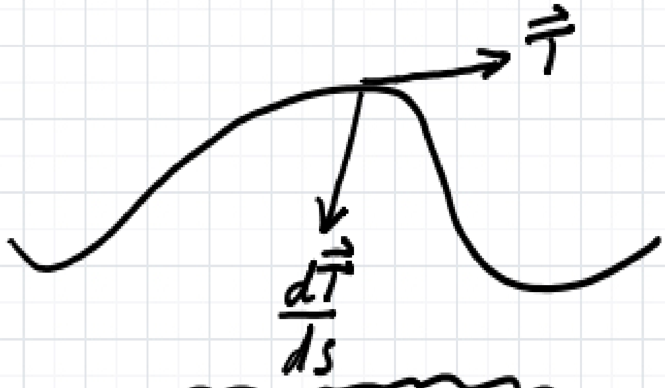
acceleration is towards origin

$$|\vec{T}| = 1$$

$$\Rightarrow \boxed{K = \frac{1}{R}} \qquad \text{as } R \to \infty \qquad K \to 0$$

$\boxed{\text{large Radius circle looks straight!}}$

PRINCIPAL UNIT NORMAL



points into bend

Make unit vector by dividing by length

$$\vec{N} = \frac{d\vec{T}}{ds} \Big/ \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{\kappa} \frac{d\vec{T}}{ds}$$

$\boxed{\text{PRINCIPAL UNIT NORMAL}}$

<u>Note</u>

inverse speed

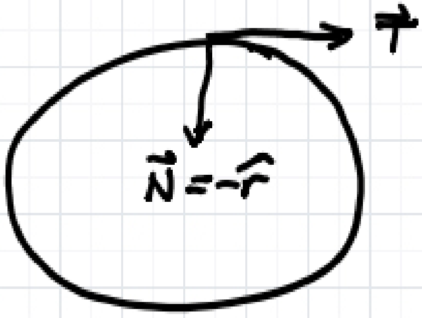
$$\vec{N} = \frac{d\vec{T}}{ds} \Big/ \left| \frac{d\vec{T}}{ds} \right| = \frac{d\vec{T}}{dt} \frac{dt}{ds} \Big/ \left| \frac{d\vec{T}}{dt} \right| \left| \frac{dt}{ds} \right|$$

$$= \frac{d\vec{T}}{dt} \Big/ \left| \frac{d\vec{T}}{dt} \right|$$

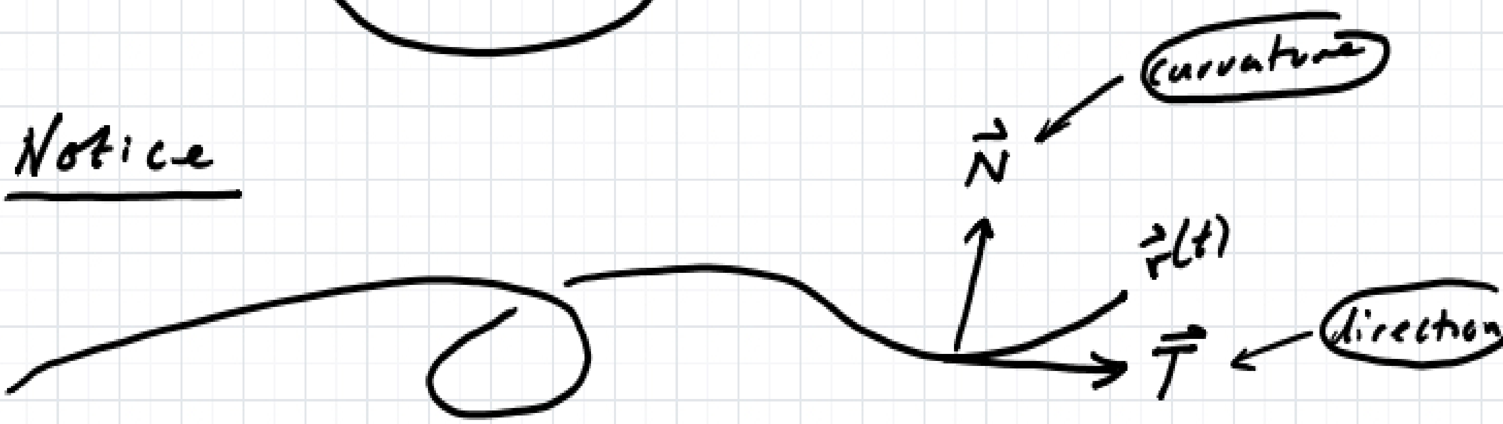
Example For circle above

$$\frac{d\vec{T}}{dt} = -\frac{1}{R}\vec{r} \Rightarrow \frac{d\vec{T}}{dt}\Big/\left|\frac{d\vec{T}}{dt}\right| = -\frac{1}{R}\vec{r}\Big/\frac{1}{R}|\vec{r}|$$

$= -\hat{r}$ unit inpointing vector



Notice



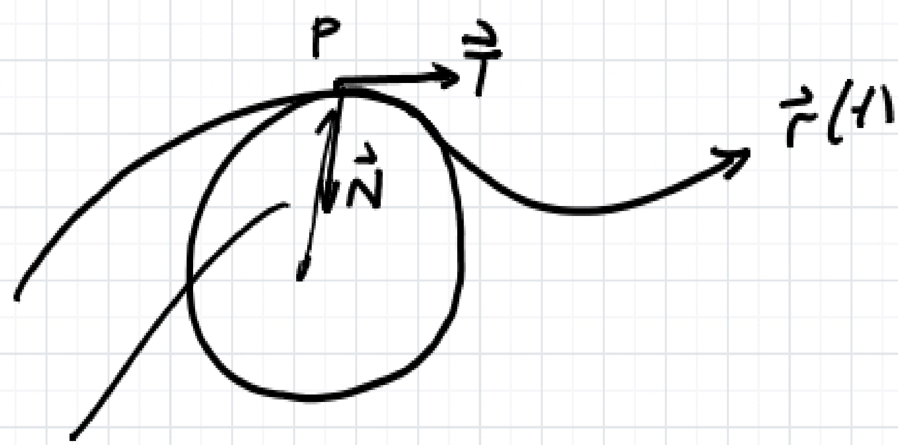
so far have found 2 unit vectors associated with path. There is a 3rd

$\hat{B} = \vec{T} \times \vec{N}$ - unit binormal

- measures twisting (see 13.5)

CIRCLE OF OSCULATION

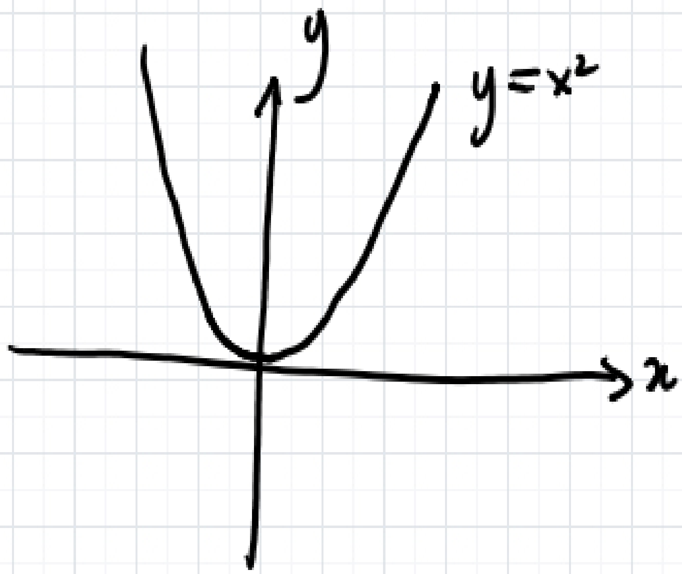
Momentarily, a curve with curvature K, tangent vector $\vec{T}$ and principal normal $\vec{N}$ will feel like motion on a circle:



radius $\frac{1}{K}$ (recall $K = \frac{1}{R}$ for radius R circle)

This circle is called the <u>circle of curvature</u> or osculating circle at P.

<u>Example</u> Find the osculating circle to the parabola $y = x^2$ at the origin



Choose parameterization $\vec{r} = t\,i + t^2 j$

($x = t$ say, origin is $t = 0$)

$$\dot{\vec{r}} = i + 2tj \Rightarrow |\dot{\vec{r}}| = |\vec{v}| = \sqrt{1+4t^2}$$

$$\vec{T} = \frac{i+2tj}{\sqrt{1+4t^2}} \Rightarrow K(t) = \frac{|\dot{\vec{T}}|}{|\dot{\vec{r}}|} = \left| \frac{8t\left(-\frac{1}{2}\right)}{(1+4t^2)^{3/2}}(i+2t) + \frac{2j}{\sqrt{1+4t^2}} \right| \Big/ \sqrt{1+4t^2} \Rightarrow K(0) = \left|\frac{2j}{1}\right| \Big/ 1 = 2$$

$\Rightarrow$ Radius $R = \frac{1}{2}$