LECTURE #13 LINE INTEGRALS ch 16.1

Our first example of a line integral was the arc length

$$L = \int_P^Q ds$$

This is a lot like our first area integral

$$A = \int_R dA$$

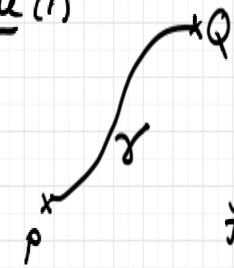
But just like we could consider $\int_R f\,dA$

for some function $f : R \longrightarrow \mathbb{R}$

we can also compute $\int_P^Q f\,ds$.

The function f could be defined in different ways.

Example (i)

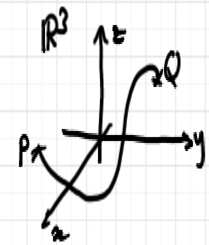


γ some path

$f : \gamma \longrightarrow \mathbb{R}$

compute $\int_\gamma f \, ds$

Example (ii)



$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$
defined on all of space

$\Rightarrow$ f also defined on γ

compute $\int_\gamma f \, ds$ (but many choices of γ exist!)

To compute $\int_\gamma f\,ds$ we choose

a parameterization $\vec{r}(t)$ for γ.

Then find $f(\vec{r}(t))$ now depending on t

and write

$$\int_\gamma f\,ds = \int_{t_P}^{t_Q} f(\vec{r}(t))\,\frac{ds}{dt}\,dt$$

$|\vec{v}|$ the speed

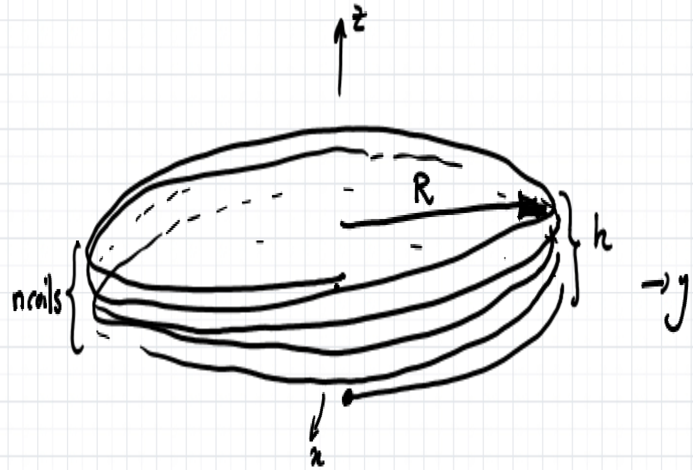
When $f = 1$, unit function, we

simply recover our old formula for

the arc length!

<u>Example</u> The density of a radius R, height h spring with n coils varies steadily from zero to ρ mass units/unit length from top to bottom. Find its ~~total~~ mass.

<u>Picture</u>



<u>Parameterization</u>

$$\vec{r}(t) = R(\cos t\, i + \sin t\, j) + \frac{ht}{2n\pi} k$$

$$0 \leq t \leq 2n\pi$$

Find density function $f(\vec{r}(t))$

$$f(\vec{r}) = \frac{\rho s}{L}$$

s = arc length

L = total length.

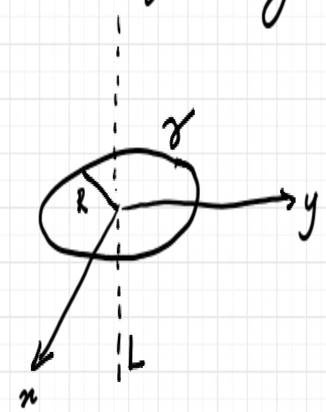
Total mass $M = \int \frac{\rho s}{L}\, ds = \frac{\rho}{L} \int_0^{s^2 = L^2} \frac{1}{2}\, d(s^2)$

$$= \frac{\rho}{2L} \, s^2 \Big|_0^{s^2 = L^2} = \frac{\rho L^2}{2L} = \frac{\rho L}{2}$$

<u>Exercise</u> compute the length

$L = \int ds$ of the above coil.

Example: Compute the moment of inertia of the hoop $x^2+y^2=R^2$ with constant density ρ mass units/unit length about its axis of symmetry



$$I_L = \int_\gamma ds\, \rho\, (x^2+y^2) = \rho R^2 \int_\gamma ds$$

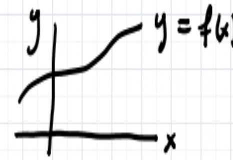
$$= 2\pi\rho R^3 = \underbrace{[2\pi R\rho]}_{M} R^2 = MR^2$$

VECTOR FIELDS (ch 16.2)

So far we studied functions

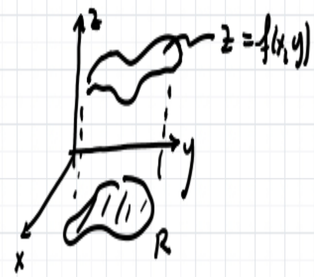
(i) $f : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto f(x)$



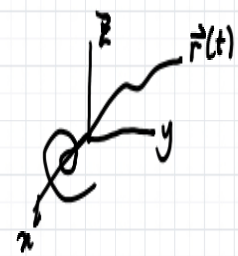
(ii) $f : R \longrightarrow \mathbb{R}$

$(x,y) \longmapsto f(x,y)$



(iii) $f : \mathbb{R} \longrightarrow \mathbb{R}^3$

$t \longmapsto \vec{r}(t)$

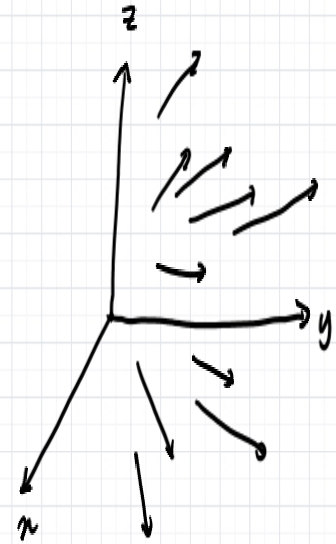


Now want to study "vector fields"

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto \vec{F}(x,y,z)$

Example The Wind



There is a vector at every point in space that measures the velocity (= direction & speed) of the wind.

Other examples: Electric & Magnetic fields $\vec{E}$, $\vec{B}$.