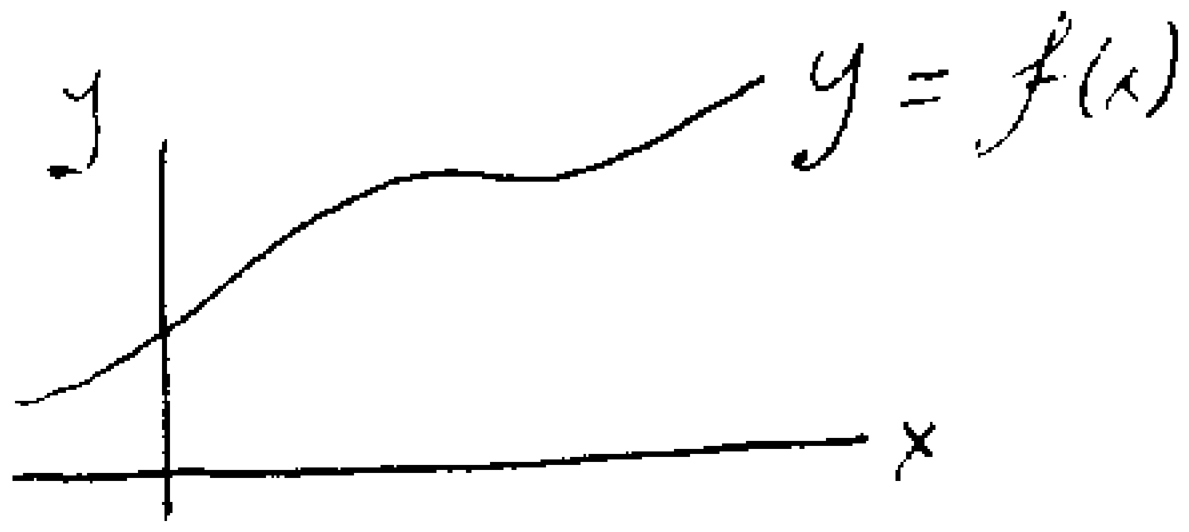
Lecture # 14 Vector Fields ch 16.2

MIDTERM II Wednesday May 13

Examples of functions studied so far

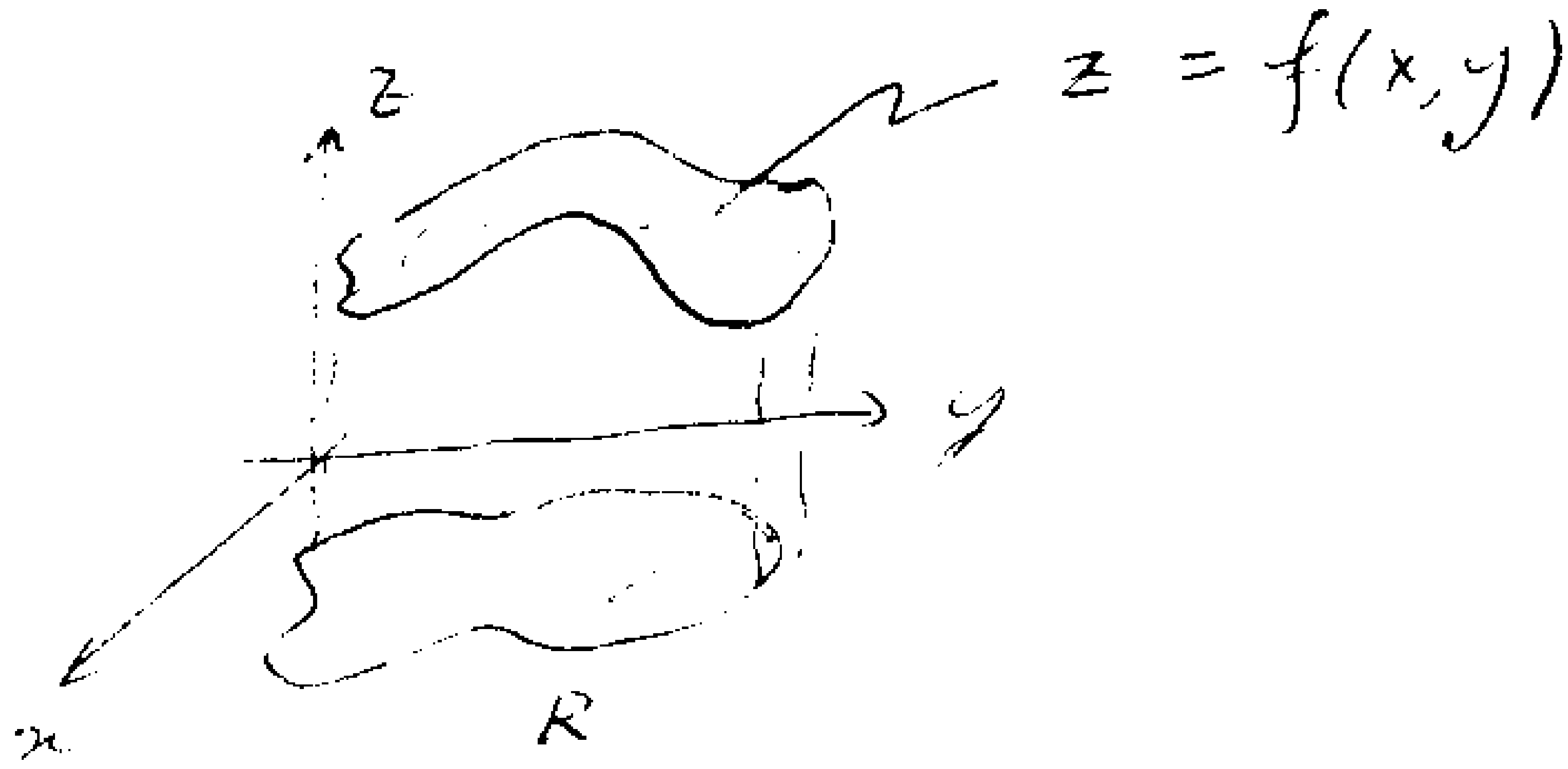
$f : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto f(x)$



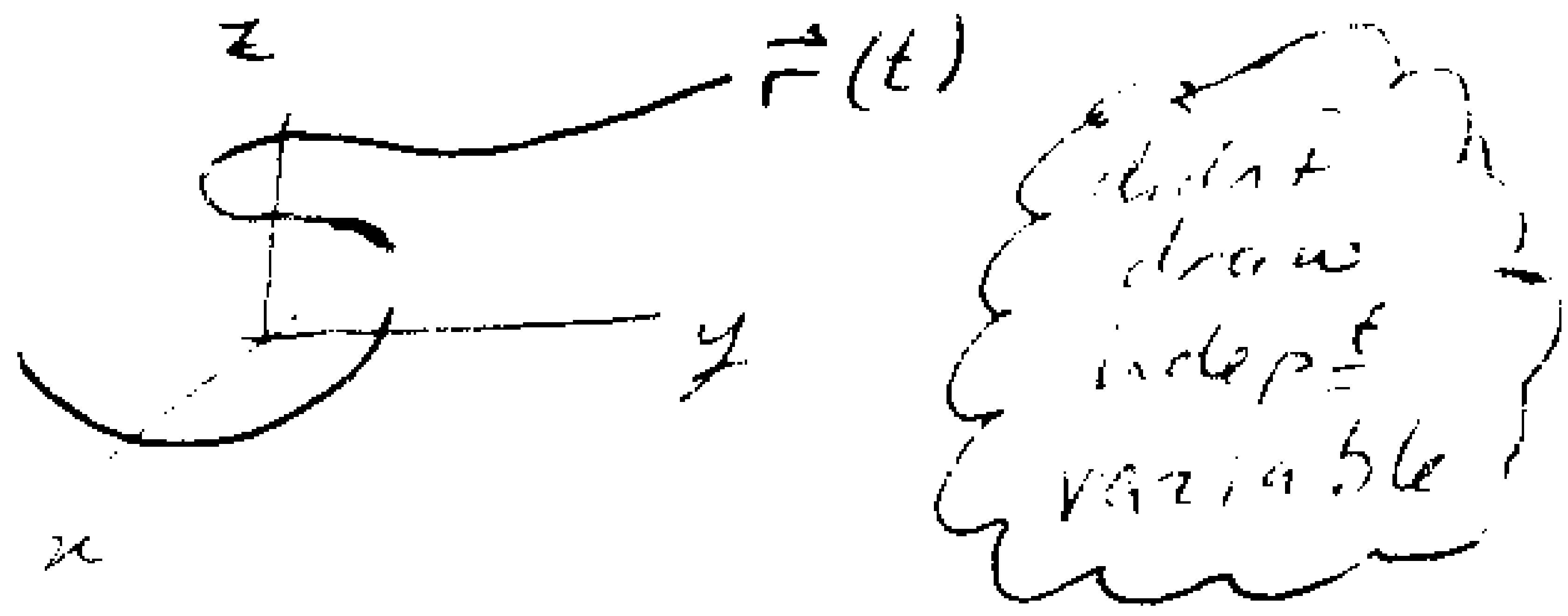
$f : R \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto f(x,y)$



$f : \mathbb{R} \longrightarrow \mathbb{R}^3$

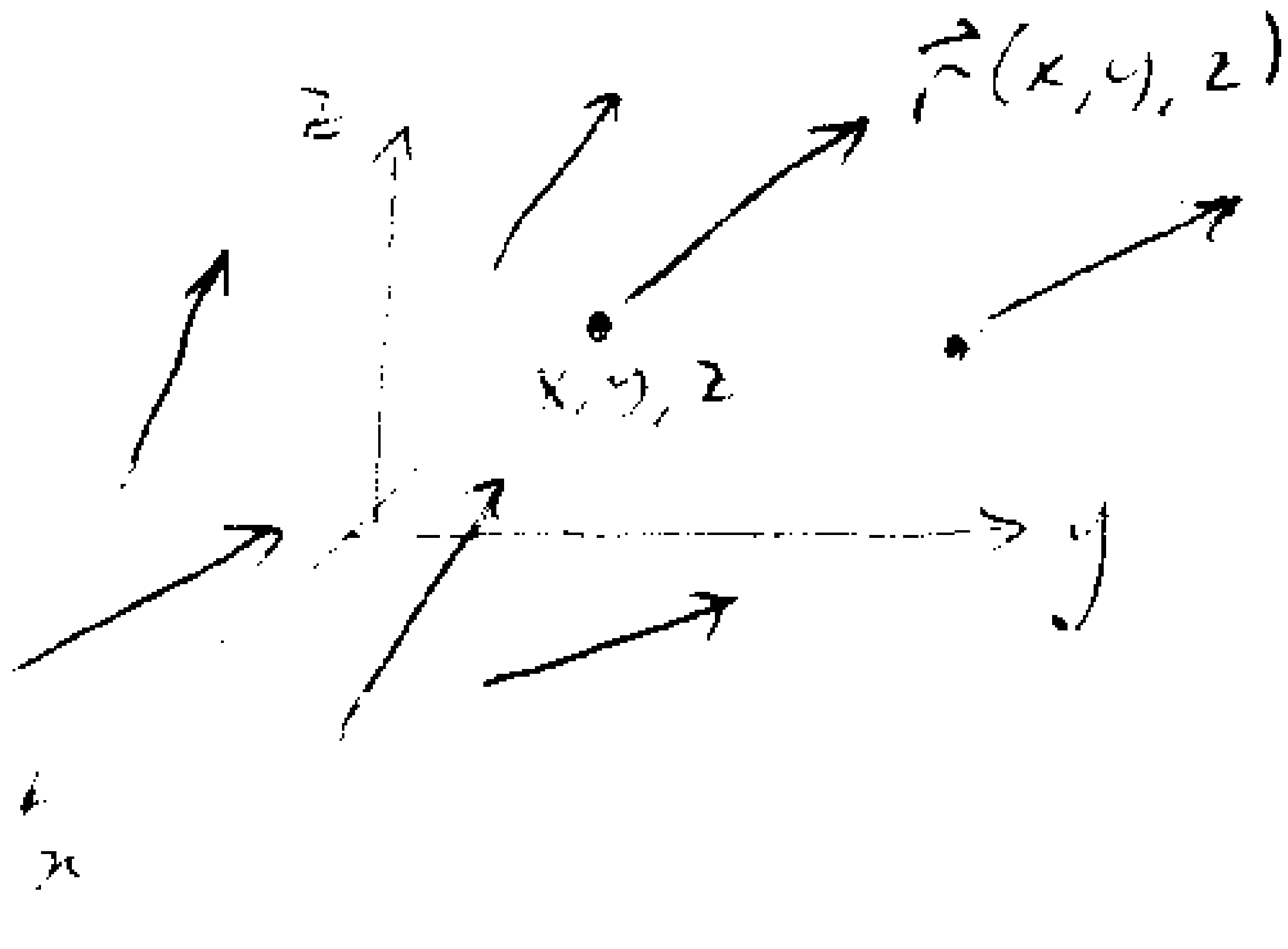
$t \longmapsto \vec{r}(t)$



Now study "vector fields"

$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto \vec{r}(x,y,z)$

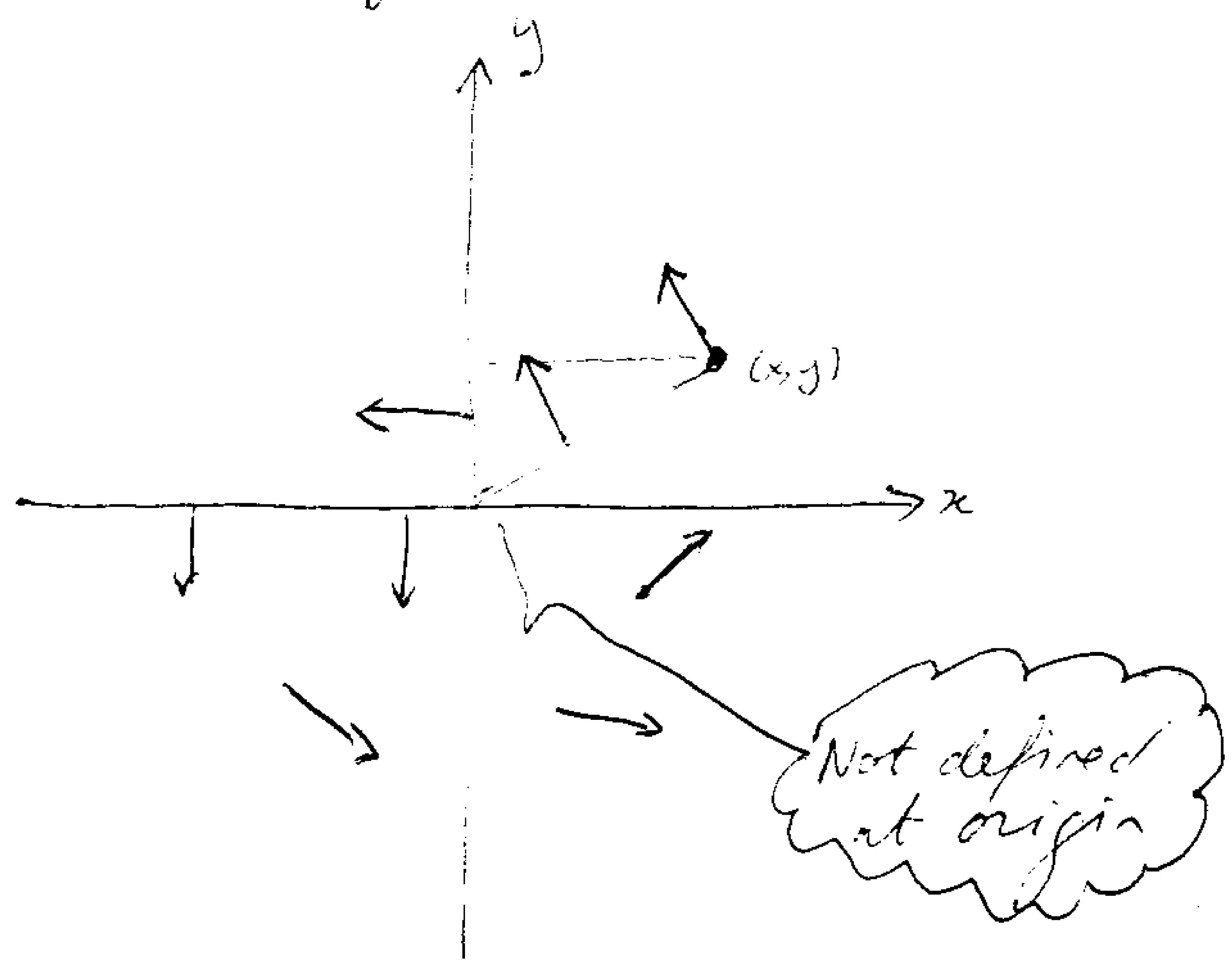


Circumferential / spin vector field

$$\vec{F} = \frac{-y i + x j}{\sqrt{x^2+y^2}},$$

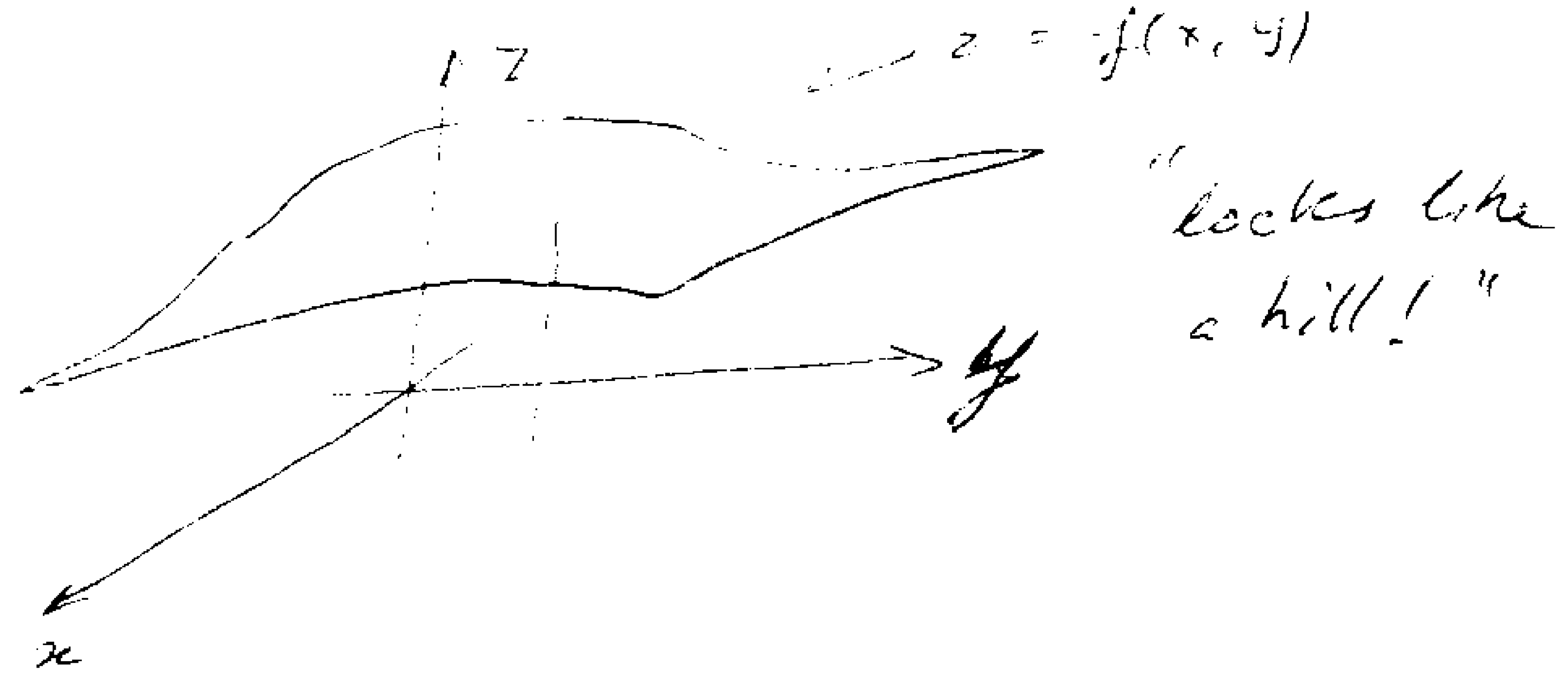
$\vec{F} \cdot (x i + y j) = 0$
perpendicular to radial vector field

$$|\vec{F}| = \frac{(-y)^2 + x^2}{(\sqrt{x^2+y^2})^2} = 1$$ unit vectors

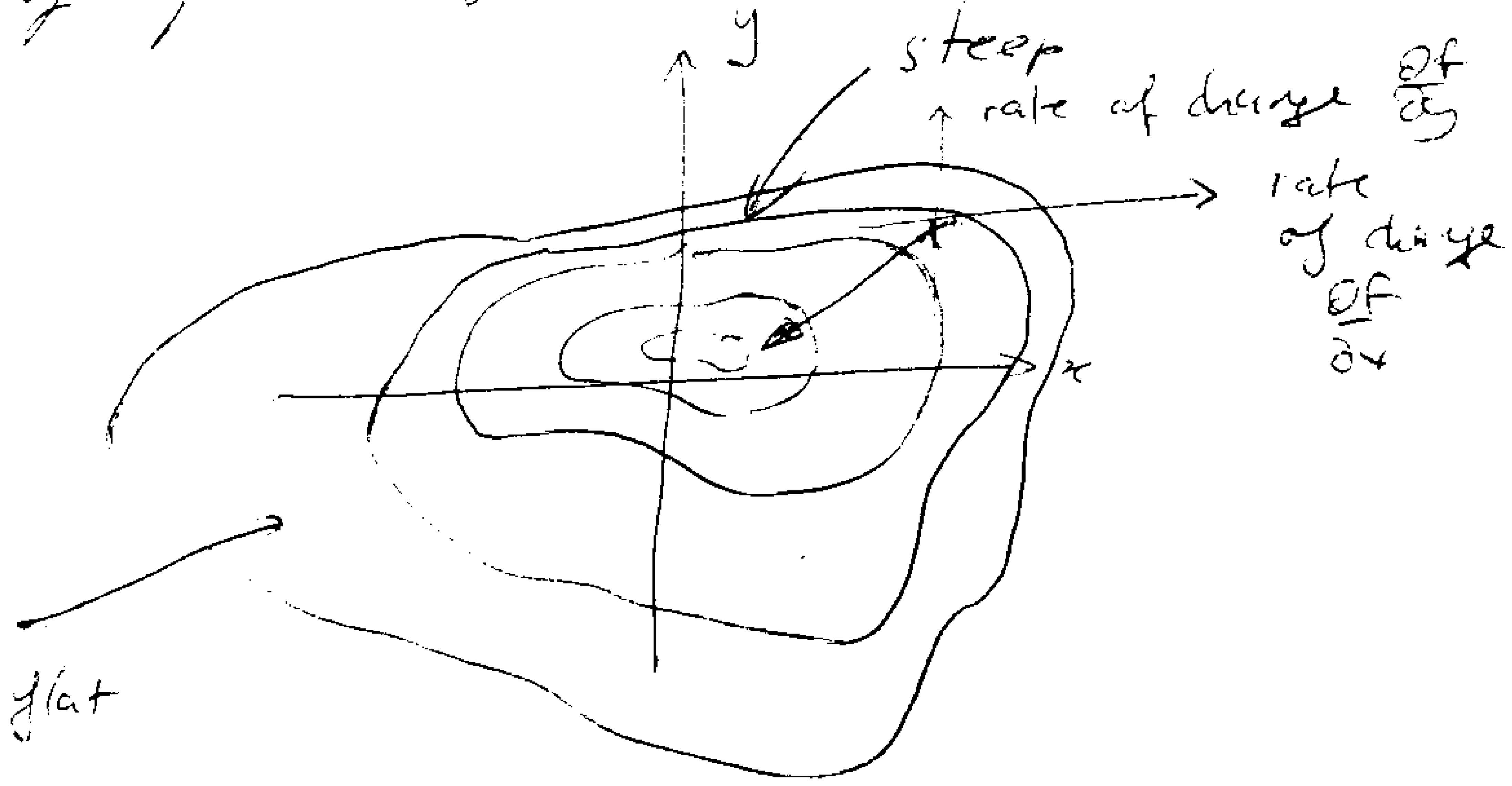


Gradient vector field

Consider $f(x,y)$



we can also depict this using contours of equal height



$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$ the gradient

is a vector pointing in the direction of increasing contours. It is longer for nearby contours.

Notice ∇ produces a vector field from a function,

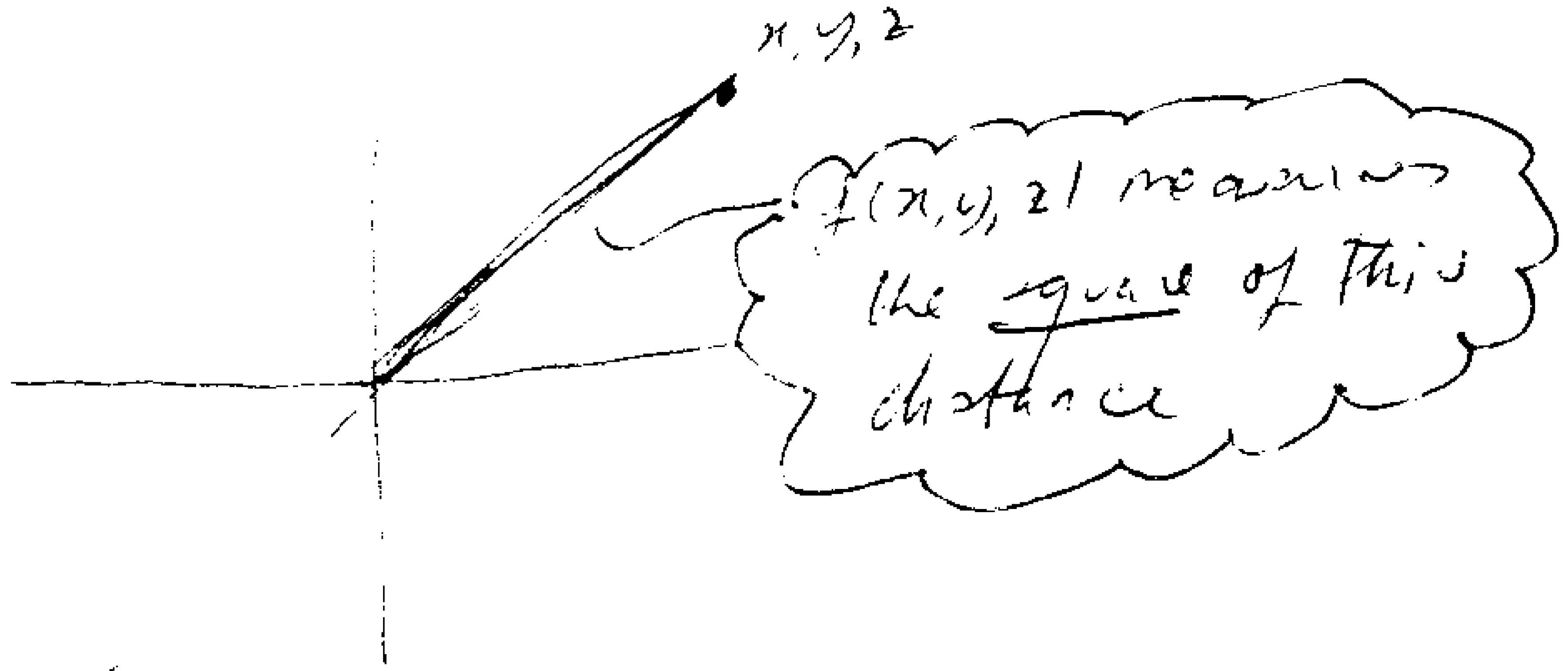
$$\boxed{\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k}.$$

is the three dimensional gradient.

It measures in which direction and how fast $f(x,y,z)$ is increasing. We call ∇f a "gradient vector field".

Example

$$f(x,y,z) = x^2+y^2+z^2 = r^2$$

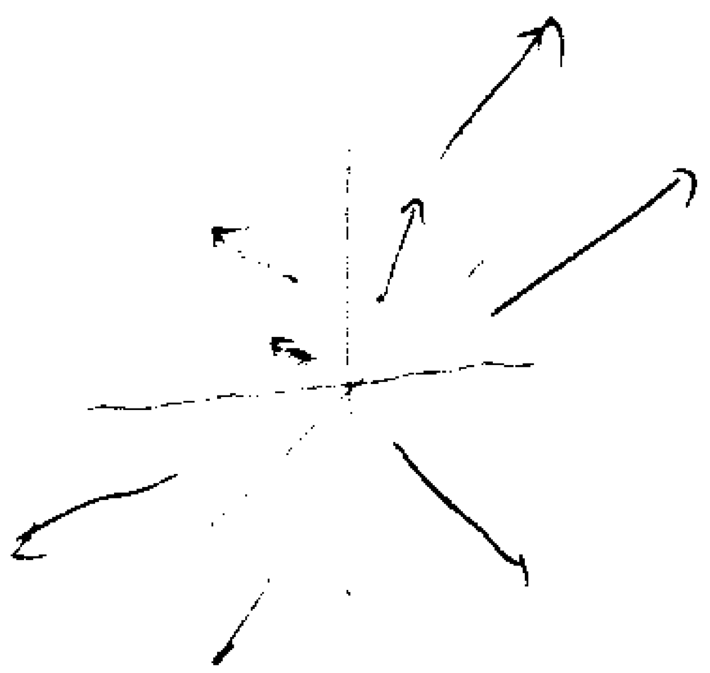


$\Rightarrow$ expect ∇f to point radially outward and get longer the further we move from the origin.

$$\nabla(x^2+y^2+z^2) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= 2ix + 2jy + 2kz$$

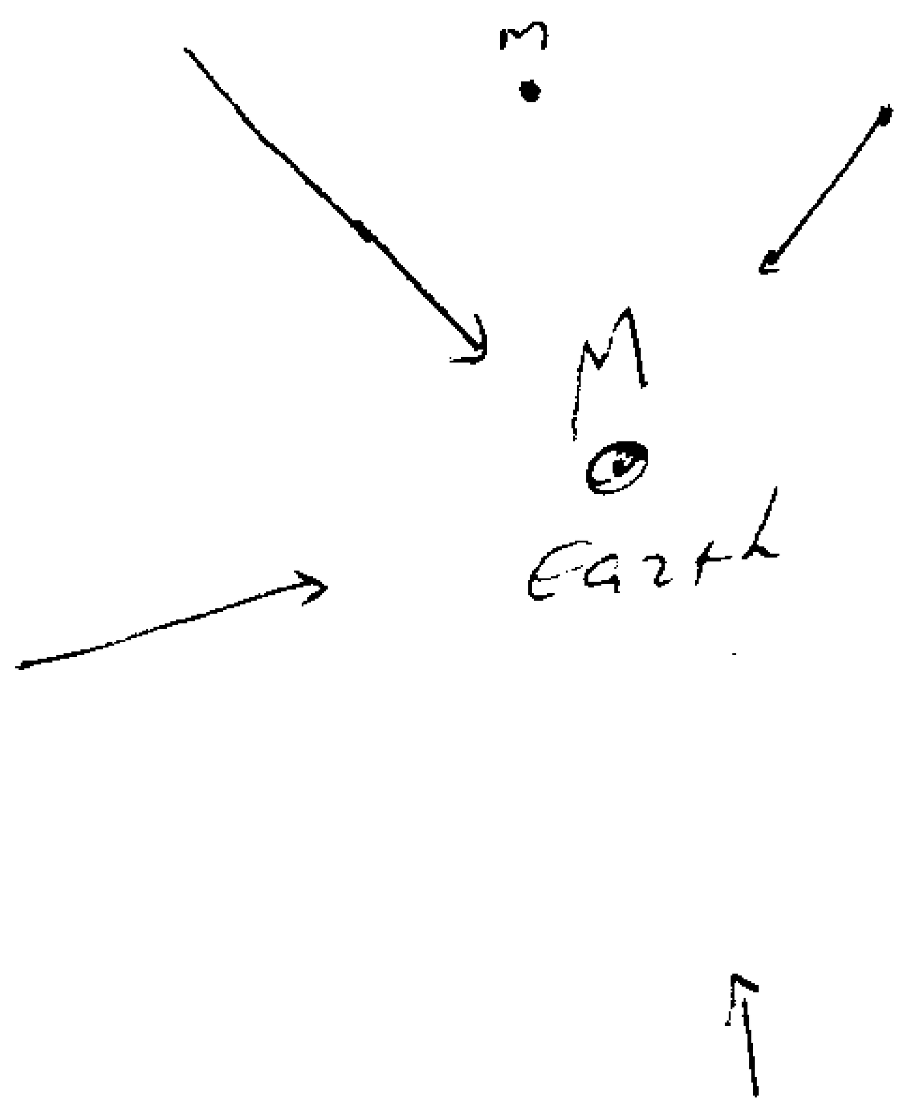
$$= 2 \times (\text{radial vector field})$$



// makes sense

Forces Force is an important example of a vector field.

Example Gravity



Newton's constant, earth mass, particle mass

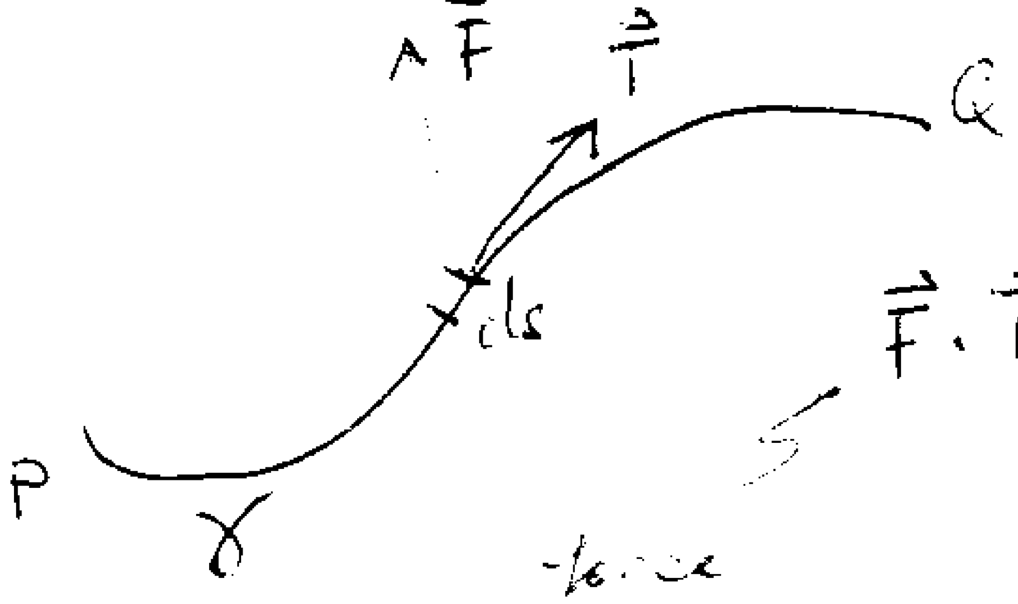
$$\vec{F} = \frac{-GMm}{x^2+y^2+z^2}\hat{r}$$

where

$$\hat{r} = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$$

unit outward pointing vector field

Work Force $\times$ distance = Work



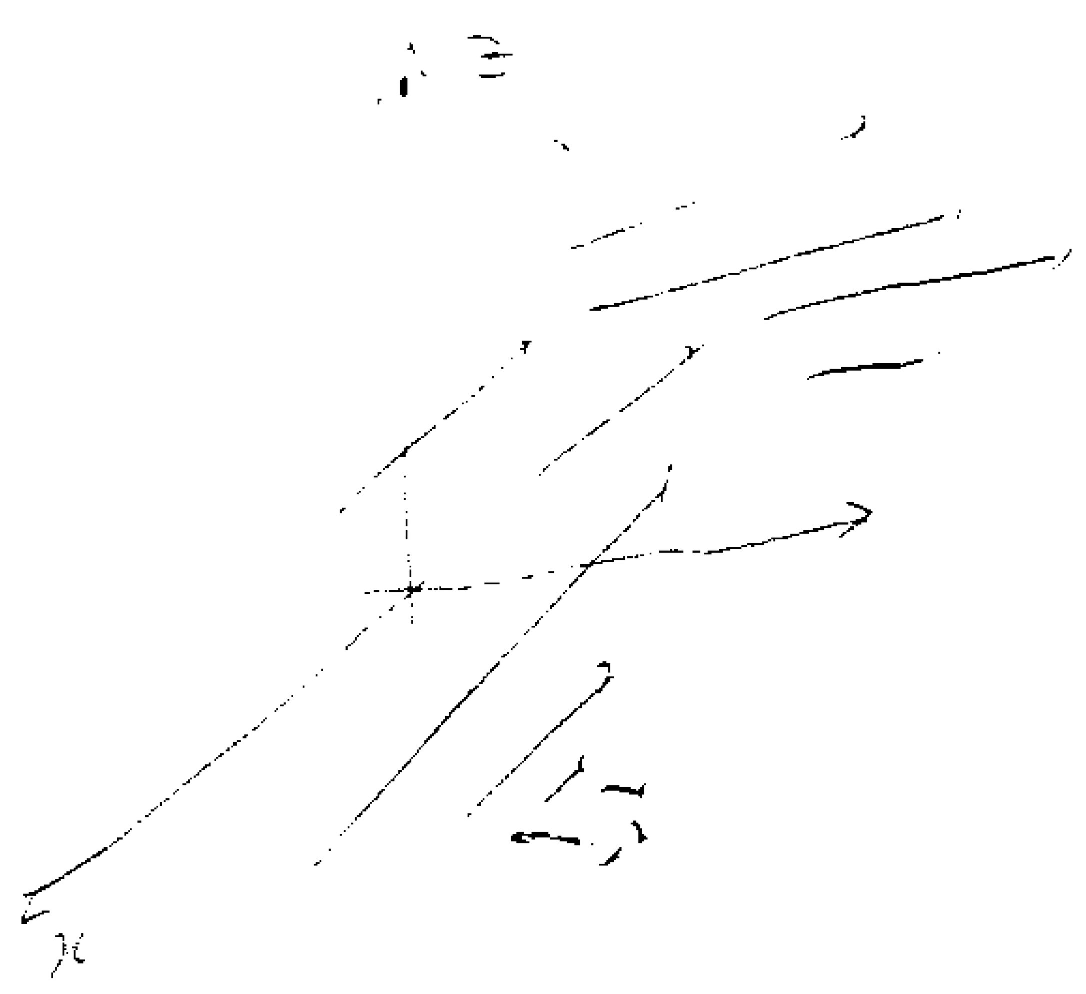
unit tangent vector

force

$\vec{F}\cdot\vec{T}\,ds$ is infinitesimal work dW

$$\boxed{W = \int_{t_P}^{t_Q} \vec{F}\cdot\vec{T}\,ds}.$$

Example The Wind!



There is a vector at every point in space that measures the velocity (= distance & speed) of the wind.

Other examples

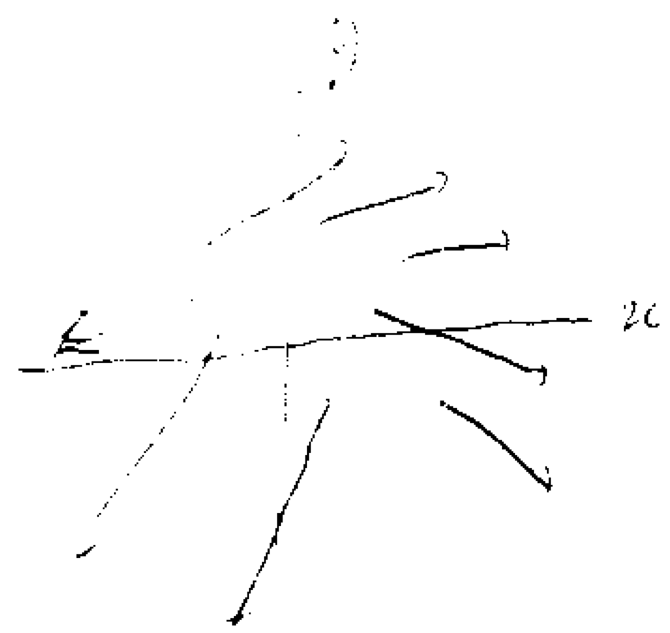
Electric & Magnetic Fields

$\vec{E}$ & $\vec{B}$

Two dimensional vector fields

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x,y) \longmapsto \vec{F}(x,y)$



Notice that a vector field $\vec{F}(x,y,z)$ is really a triple of functions of 3-variables

$$\vec{F}(x,y,z) = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$$

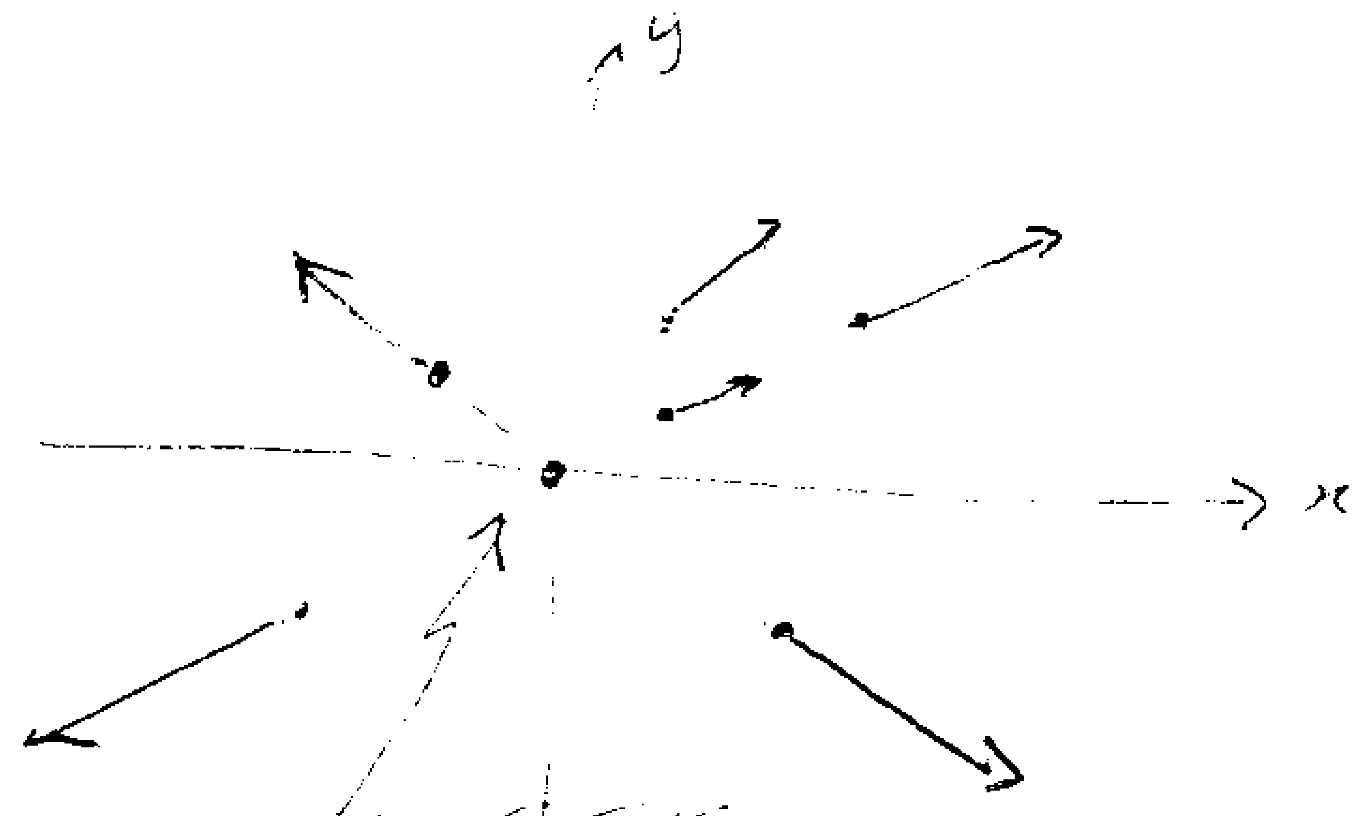
These are often called "component functions" (or just components).

Continuity, differentiability etc. of $\vec{F}(x,y,z)$ can be defined in terms of the three component functions (M, N, P).

Some explicit examples

Radial vector field

$$\vec{F} = xi + yj$$



Get zero vector at the origin