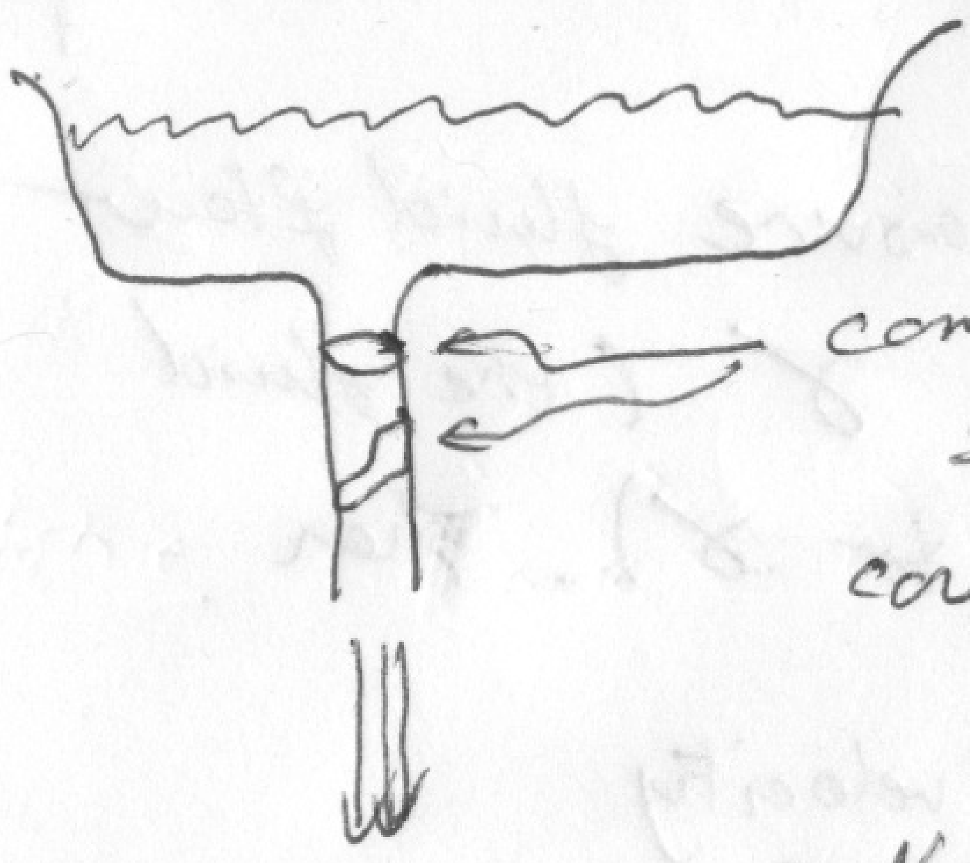
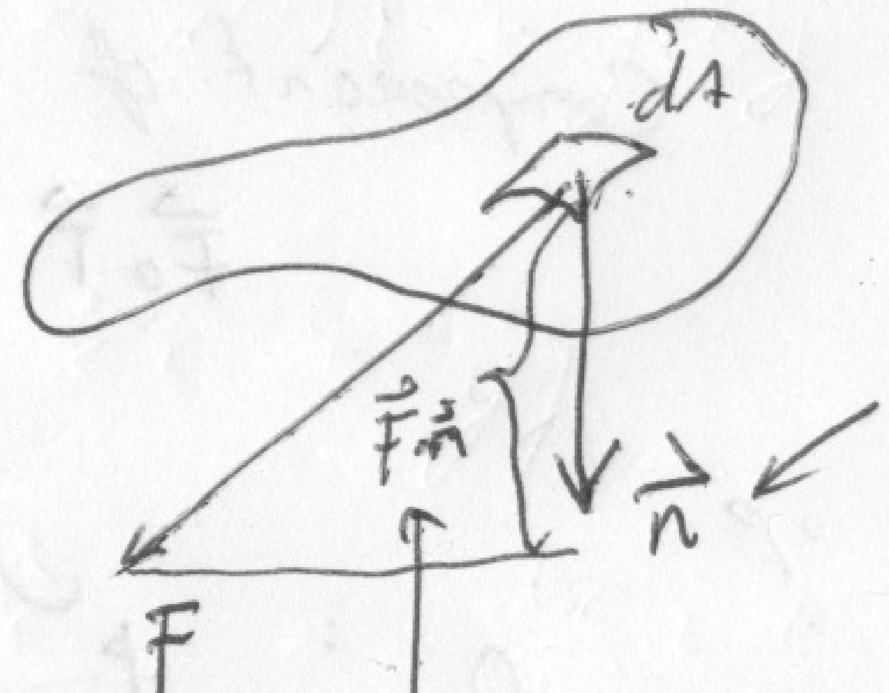
FLUX

Question How fast is water flowing out of the bath



consider some surface cove covering out let

→ Need to compute a SURFACE INTEGRAL adding up



vector normal to surface

component of $\vec{F}$ perpendicular to surface

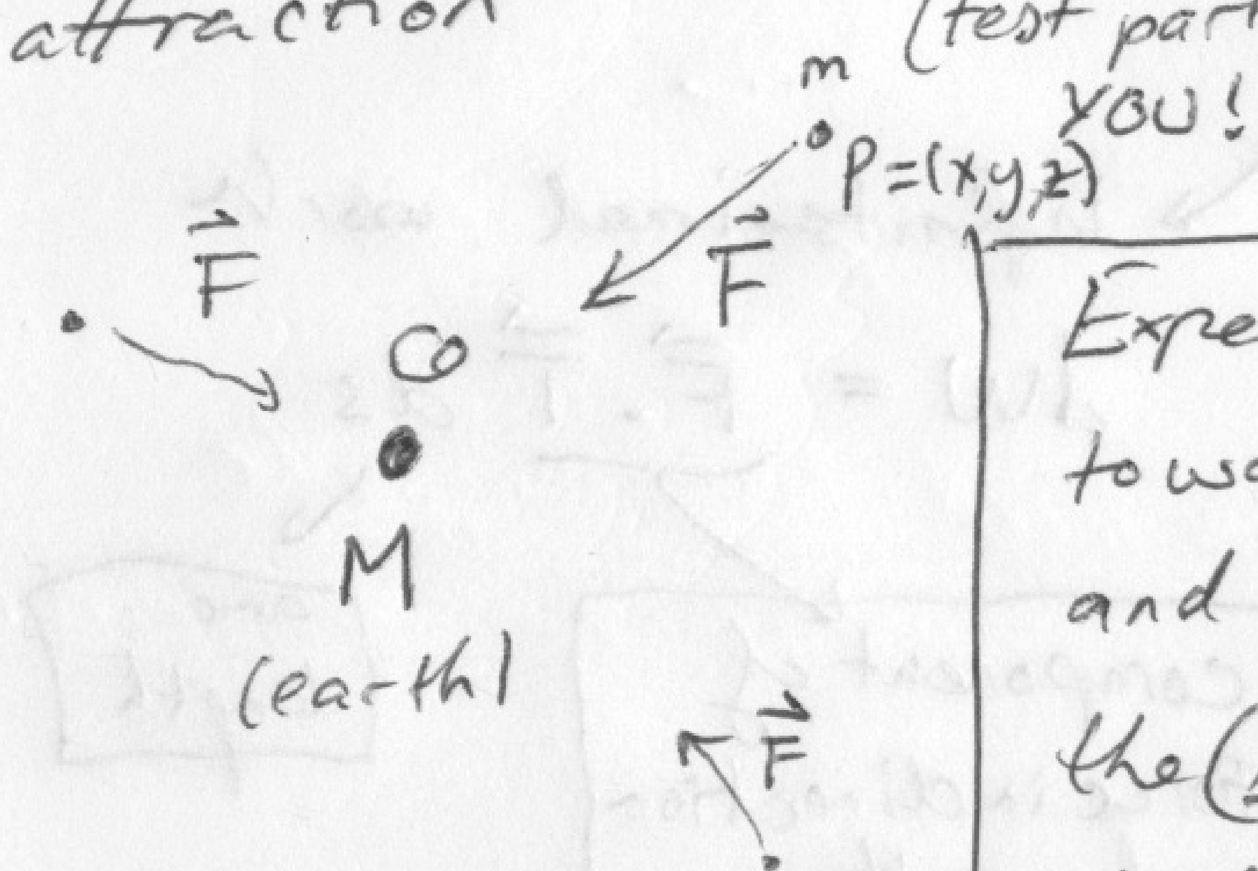
$dA\ \vec{F}\cdot\vec{n}$ = infinitesimal **FLUX**

Total flux adds these up.

We will learn surface integrals later, first do two dimensional fluid flow

Force Fields – important example of vector field – a force field $\vec{F}$ measures the force you feel due to some interaction (example gravity, electromagnetism etc...) at each point in space.

Example Gravitational force due to Earth's attraction



$\boxed{\text{Expect } \vec{F} \text{ points always towards the origin and gets weaker as the (square)}^{-1} \text{ of the distance}}$

Newton's constant

Earth mass

Particle mass

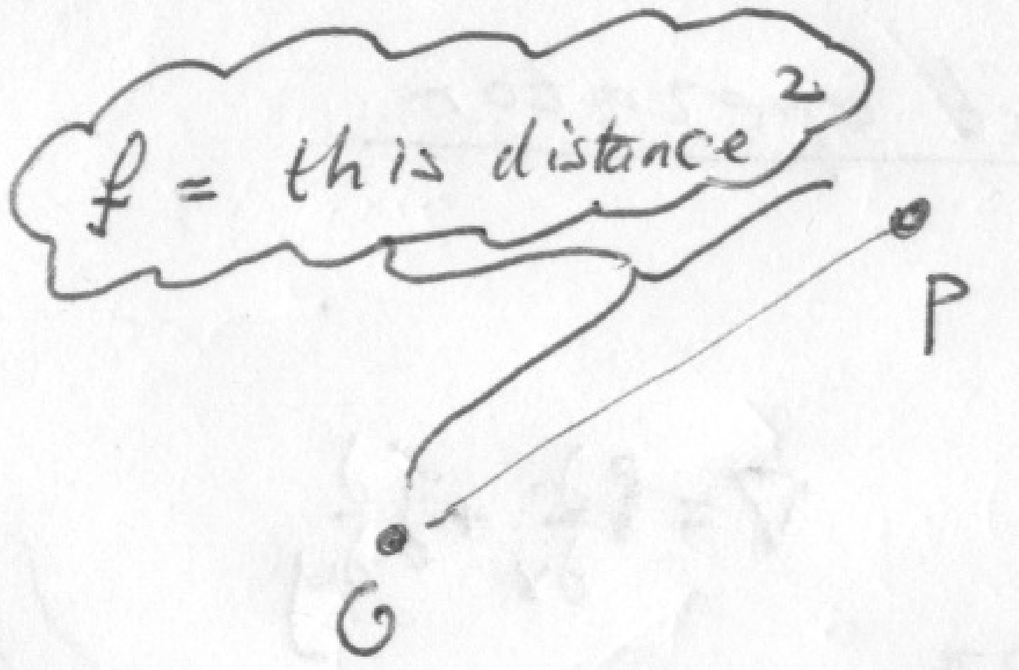
$$\vec{F} = -\frac{GMm}{x^2+y^2+z^2}\hat{r}$$

inward pointing

unit outward pointing radial vector field

$$\hat{r} = \frac{xi + yj + zk}{\sqrt{x^2+y^2+z^2}}$$

<u>Example</u> $f(x,y,z) = x^2 + y^2 + z^2 = \rho^2$



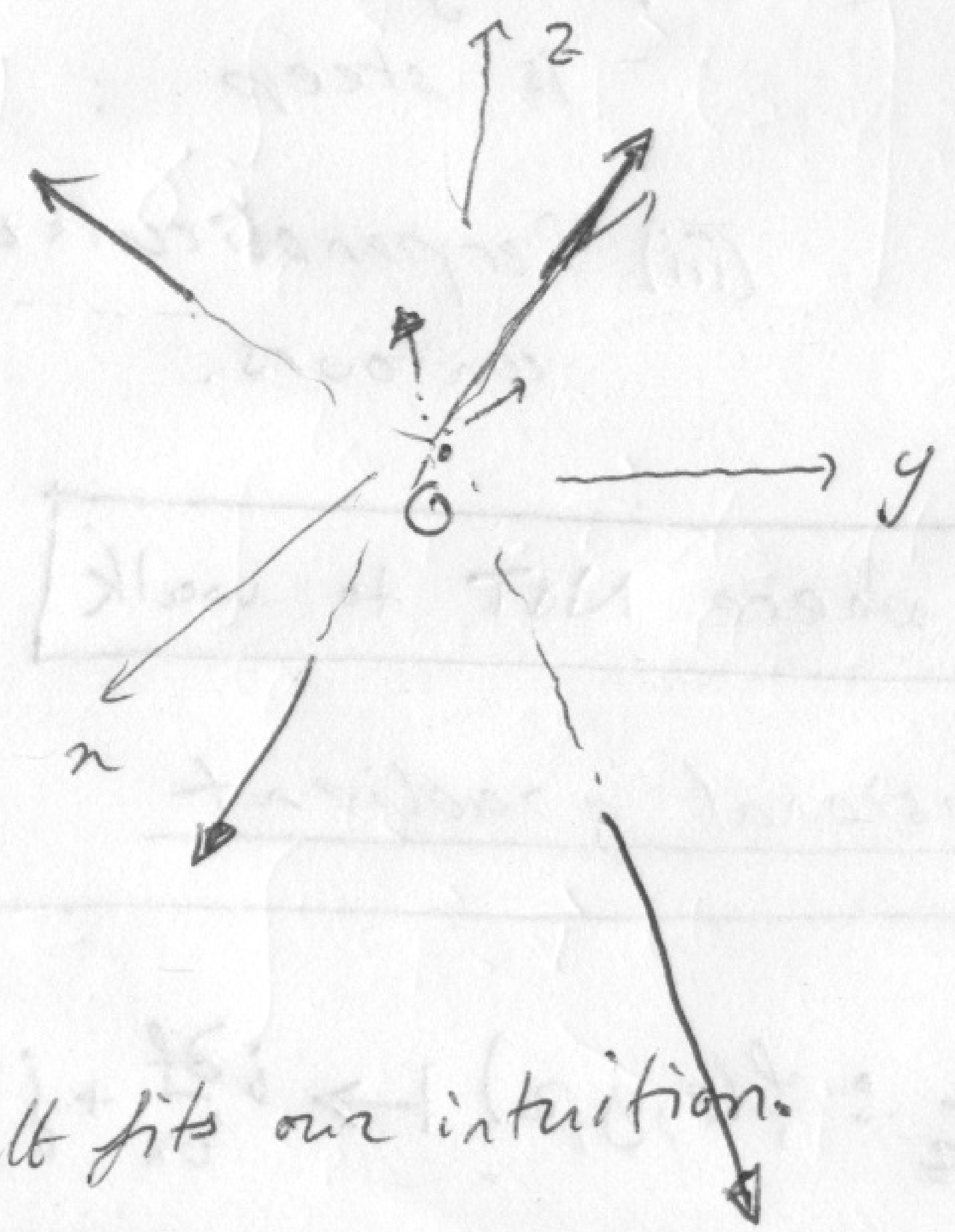
$\Rightarrow$ Expect ∇f to point <u>radially outward</u> (in direction f increases). $|\nabla f|$ should increase as we go further from the origin.

$$\nabla(x^2+y^2+z^2) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

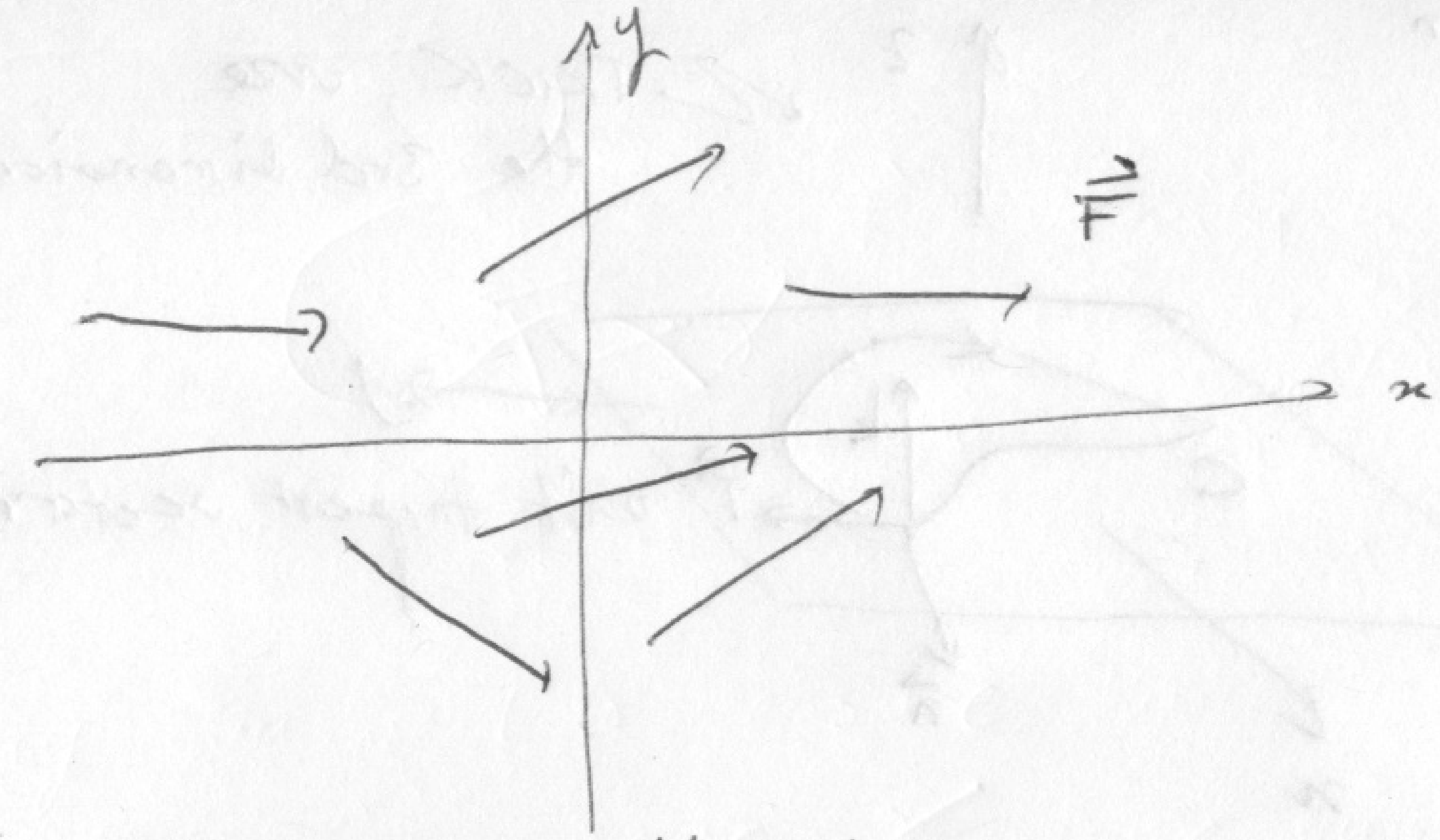
$$= 2ix + 2jy + 2kz$$

$$= 2\vec{F}$$

where $\vec{F} = xi + yj + zk$ radial vector field

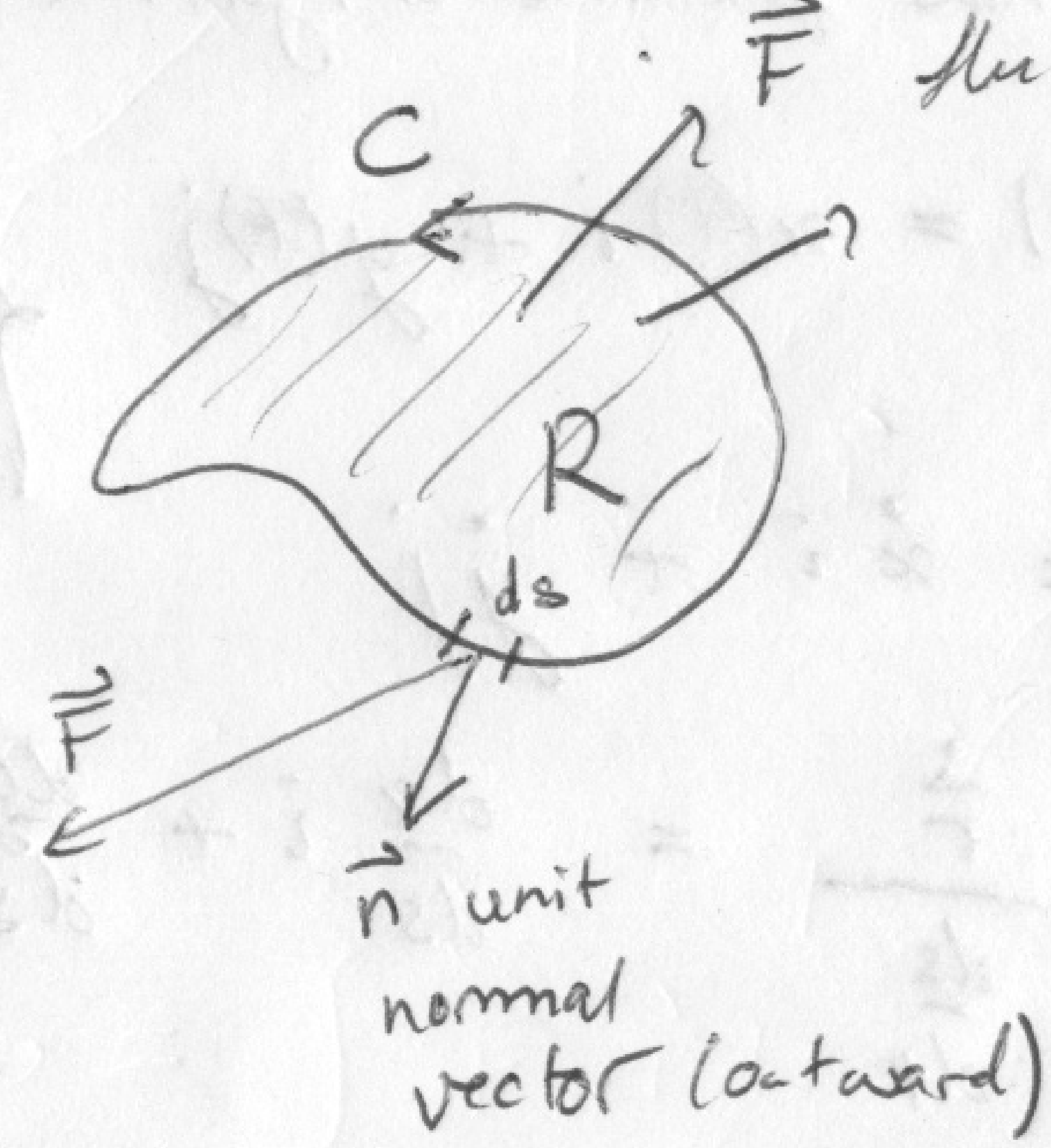


This result fits our intuition.



Fluid flow in the plane (example, looking at ocean from above)

To measure the amount of fluid leaving a region R



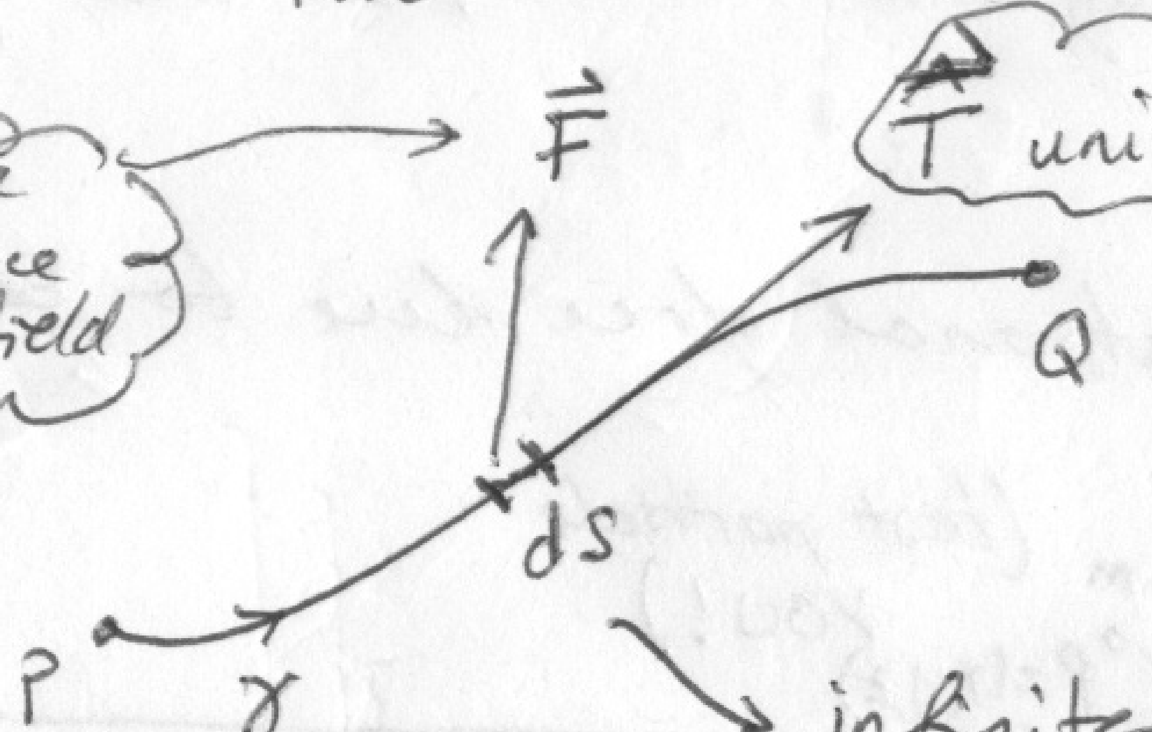
Need to add up $\vec{F} \cdot \vec{n}\, ds$

$$\text{Flux} = \oint_C \vec{F} \cdot \vec{n}\, ds$$

WORK = FORCE × DISTANCE (Energy)

$\frac{\text{mass} \times \text{distance}}{\text{time}^2}$ × distance

some force field



infinitesimal work

$$dW = \vec{F}\cdot\vec{T}\,ds$$

$\vec{F}\cdot\vec{T}$: component of force in direction of path

ds: arc length

Total work

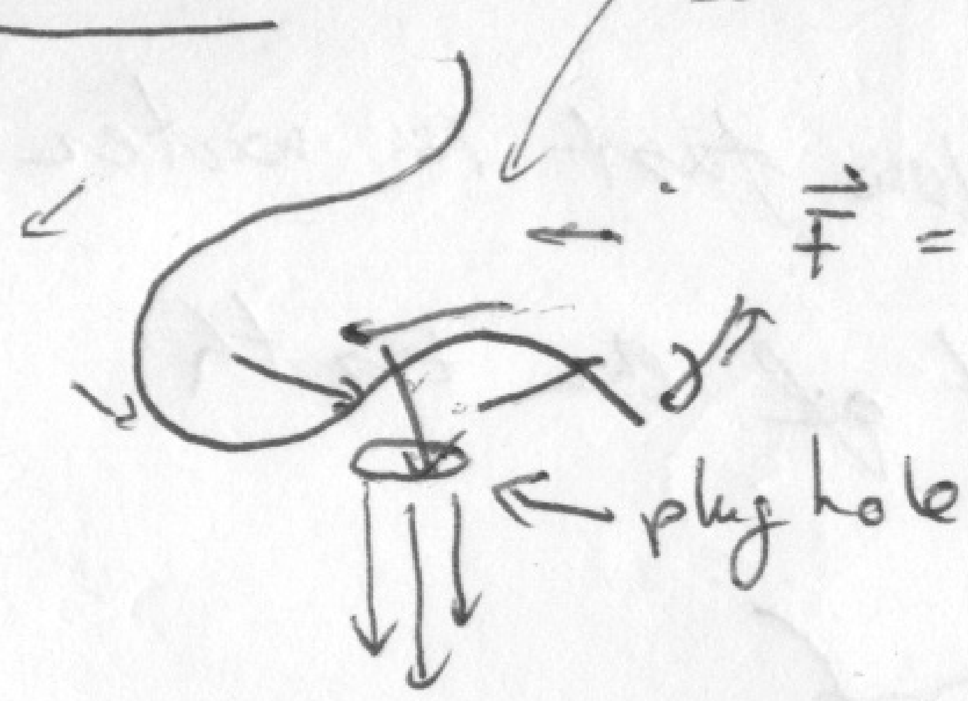
$$\boxed{W = \int_{\gamma} \vec{F}\cdot\vec{T}\,ds}$$ Line integral

$$= \int_{t_P}^{t_Q} \vec{F}\cdot\vec{T}\,\frac{ds}{dt}\,dt$$

$$= \int_{t_P}^{t_Q} \vec{F}\cdot\vec{v}\,dt$$

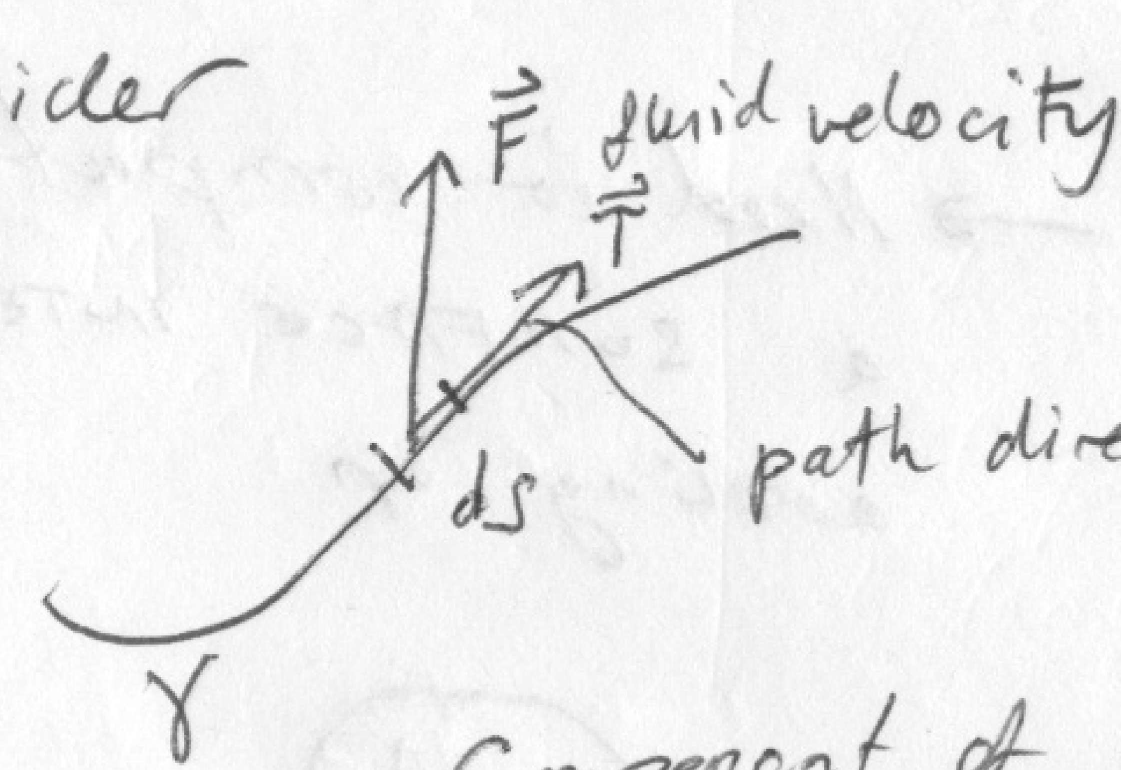
$\vec{v} = \vec{T}\,\frac{ds}{dt} = \text{velocity}$

Fluid Flow



$\vec{F}$ = velocity field
measures the water speed & direction at each point.

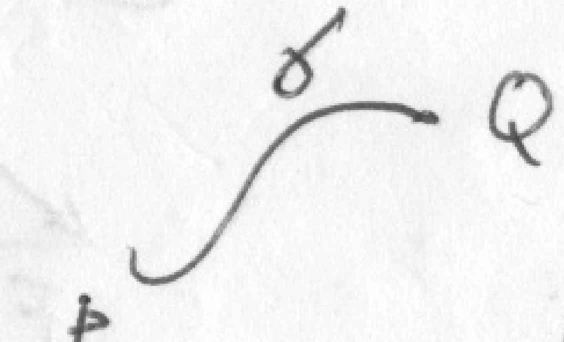
Say you want to measure fluid flow along some curve γ (the fluid need not move ∥ to γ). Then consider



Component of $\vec{F}$ along path

$\vec{F}\cdot\vec{T}$

Add these up



$$\text{Flow} = \int_P^Q \vec{F}\cdot\vec{T}\,ds = \int_{t_P}^{t_Q} \vec{F}\cdot\vec{v}\,ds$$

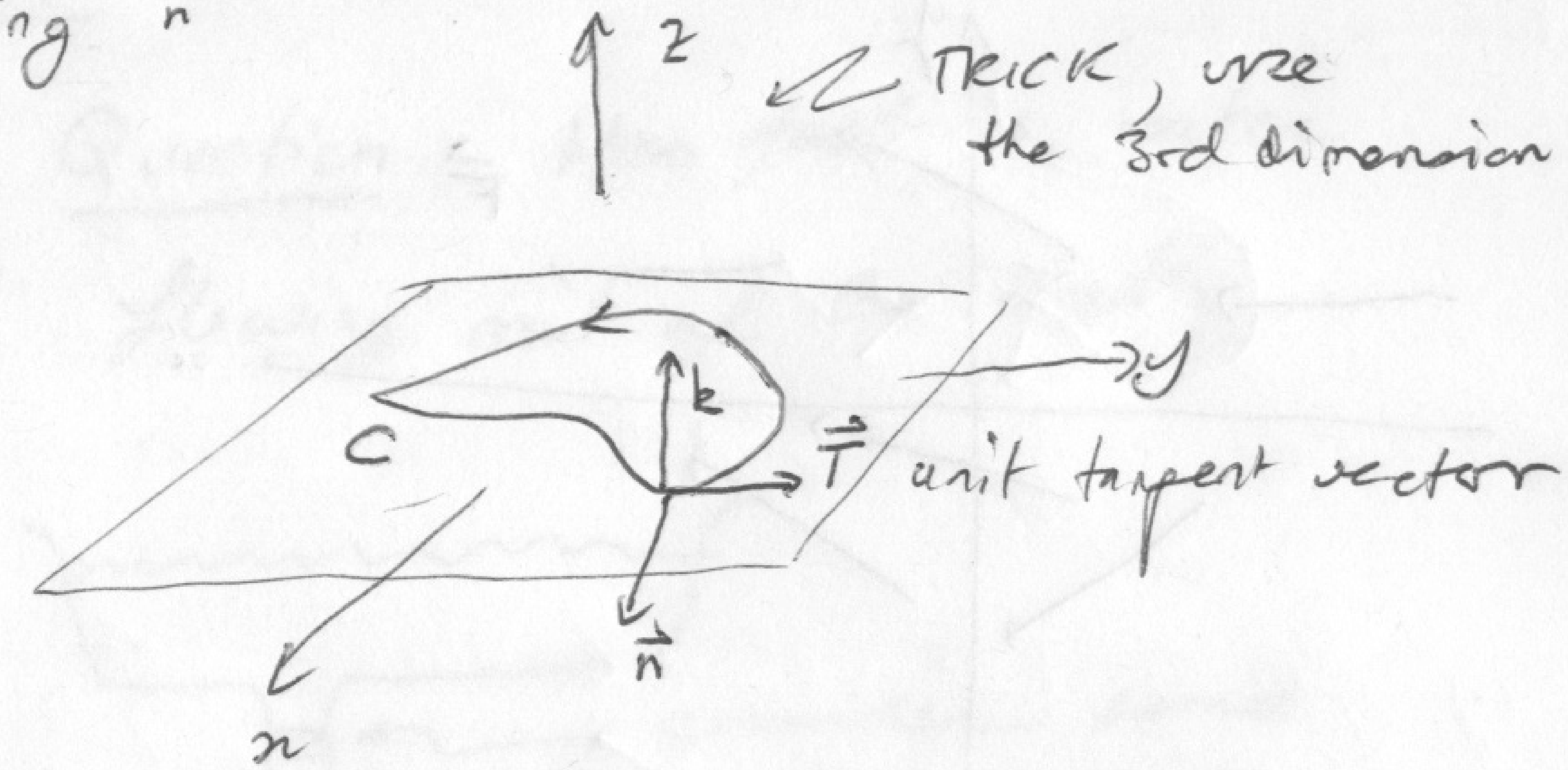
$\vec{F}$: velocity of fluid

$\vec{v}$: velocity along path γ

For a closed loop 

, we call the Flow the "circulation".

Finding $\vec{n}$



Notice $\vec{T} \times \vec{k} = \vec{n}$ (outward pointing for anticlockwise C)

$\boxed{\text{Right hand rule}}$

Now if C has parameterization

$$\vec{r}(t) = x(t)\,i + y(t)\,j \qquad \begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$$\dot{\vec{r}} = \dot{x}\,i + \dot{y}\,j$$

$$\vec{T} = \frac{\dot{\vec{r}}}{\frac{ds}{dt}} = \frac{dx}{ds}\,i + \frac{dy}{ds}\,j$$

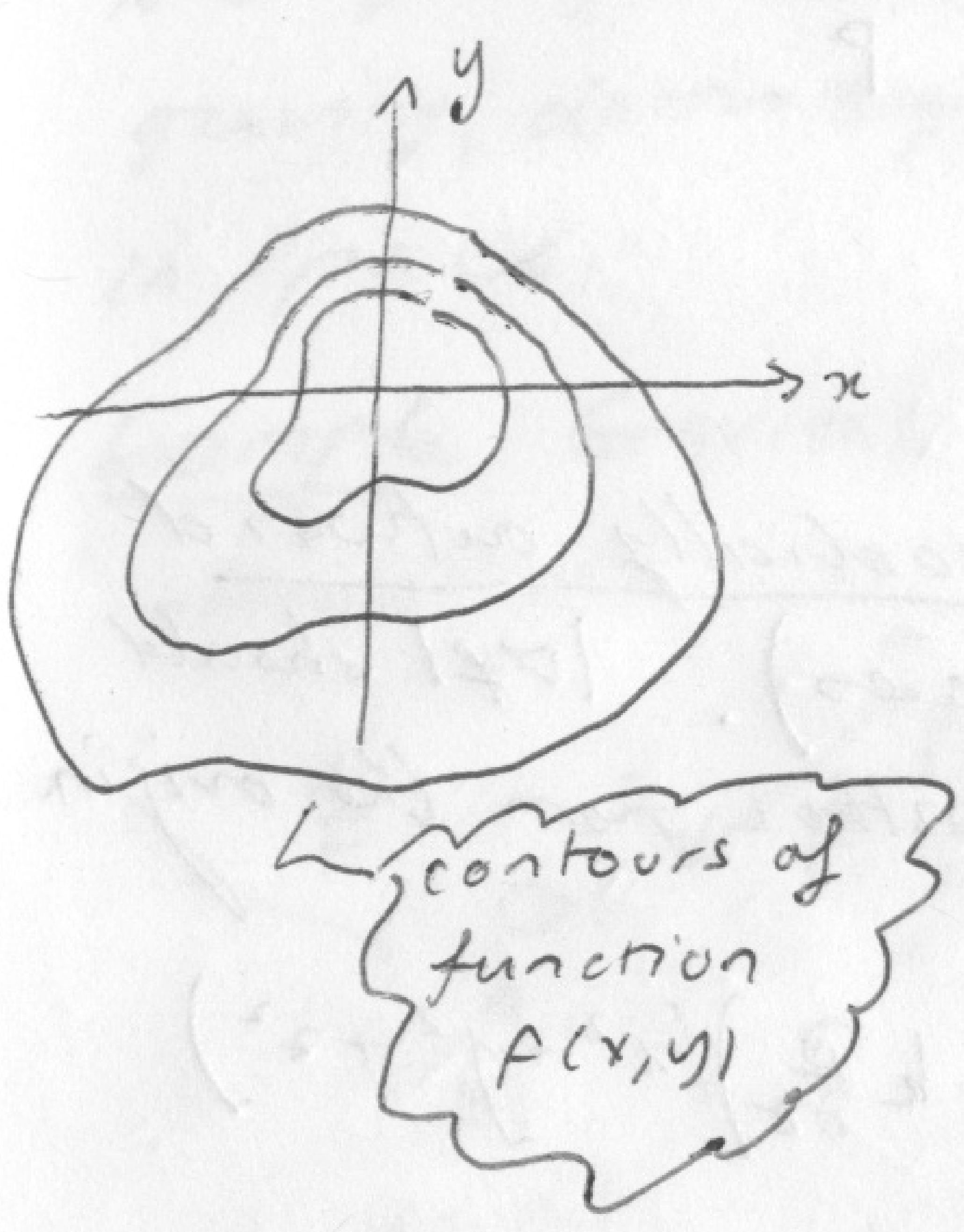
$$\Rightarrow \vec{T} \times \vec{k} = \left(\frac{dx}{ds}\,i + \frac{dy}{ds}\,j\right) \times k = +\frac{dy}{ds}\,i - \frac{dx}{ds}\,j = \vec{n}$$

If $\vec{F} = Mi + Nj$ $\quad \vec{F}.\vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$

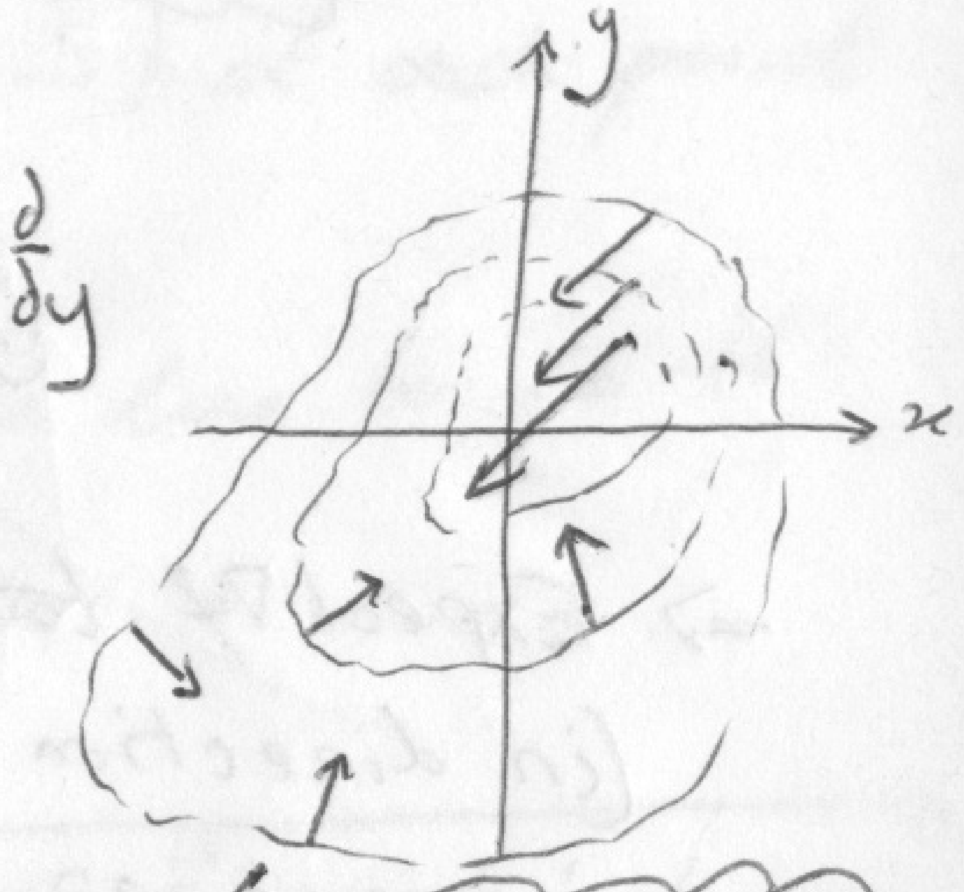
$$\Rightarrow \boxed{\text{Flux} = \oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\,dy - N\,dx}$$

Lecture # 15 VECTOR FIELDS ct. (ch 16.2)

The gradient operator



$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} \longmapsto$$



Vector field $\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y}$

∇f properties

(i) Points in direction of increasing contours

(ii) Longer when slope is steep

(iii) Perpendicular to the contours.

$\boxed{\nabla f \text{ tells you where NOT to walk}}$

Three dimensional gradient

$$\boxed{\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} : f(x,y,z) \longmapsto i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}}$$