Potential function

If $\vec{F} = \nabla f$ we call f a potential function for $\vec{F}$.

Indeed, for "suitably differentiable $\vec{F}$",

$$\boxed{\vec{F} \text{ conservative} \iff \vec{F} = \nabla f \text{ for some } f}$$

Note In physics, f is often called V

To see why this is true consider the two dimensional case

$$\vec{F}(x,y) = \nabla f(x,y) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) f(x,y)$$

$$= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y}$$

$$d\vec{r} = d(xi + yj) = (i\,dx + j\,dy)$$

$$\Rightarrow \int_P^Q d\vec{r}\cdot\vec{F} = \int_P^Q \left(dx\frac{\partial f}{\partial x} + dy\frac{\partial f}{\partial y}\right)$$

$$= \int_{t_P}^{t_Q} dt\left[\dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y}\right]$$

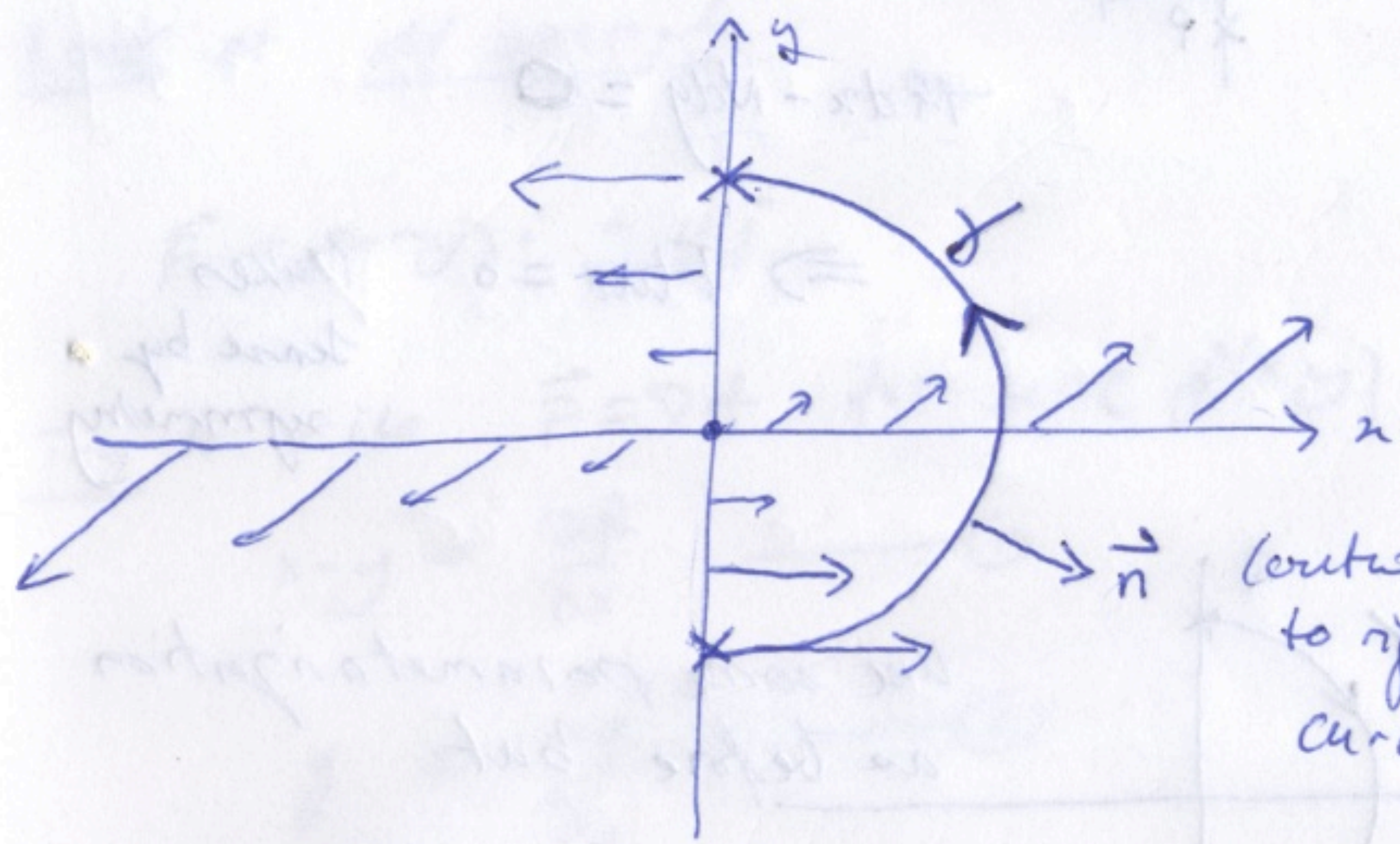
choose parameterization $x = g(t)$ $y = h(t)$ of some path $\vec{r}(t)$ b/w P & Q

Example The flux of

$$\vec{F} = (x-y)i + xj \Rightarrow \begin{cases} M = x-y \\ N = x \end{cases}$$

across a ~~curve~~ connecting

$$P = (0,-1) \quad \text{and} \quad Q = (0,1)$$



(outward to right of curve)

Lets try different γ

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$dx = d(\cos t) = -\sin t\,dt$$

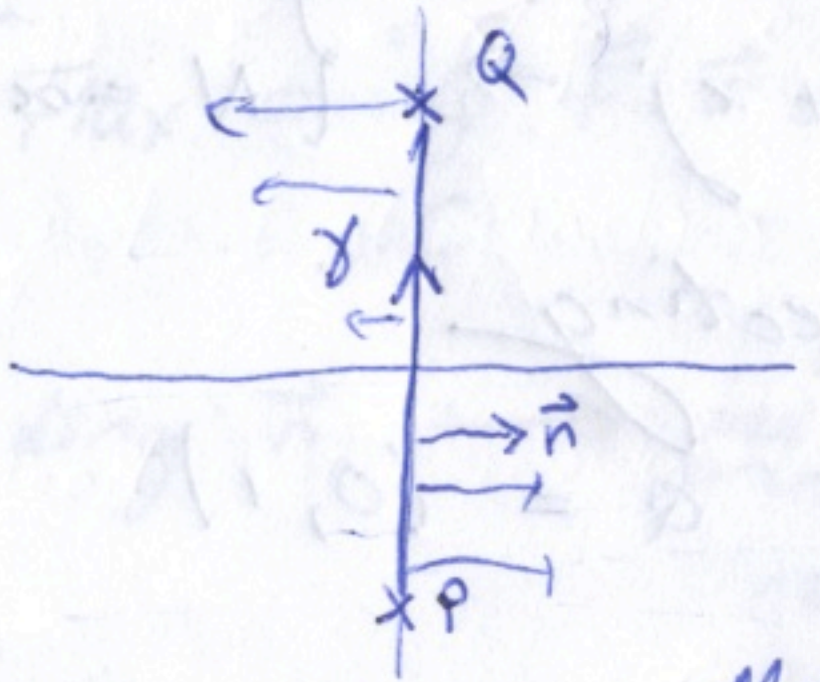
$$dy = d(\sin t) = \cos t\,dt$$

$$\Rightarrow M\,dy - N\,dx = \left[(\cos t - \sin t)\cos t + \cos t \sin t\right]dt$$

$$= \cos^2 t = \tfrac{1}{2} + \tfrac{1}{2}\cos 2t$$

$$\text{Flux} = \int_{-\pi/2}^{\frac{\pi}{2}} \cos^2 t = \frac{\pi}{2}$$

Try some other curves



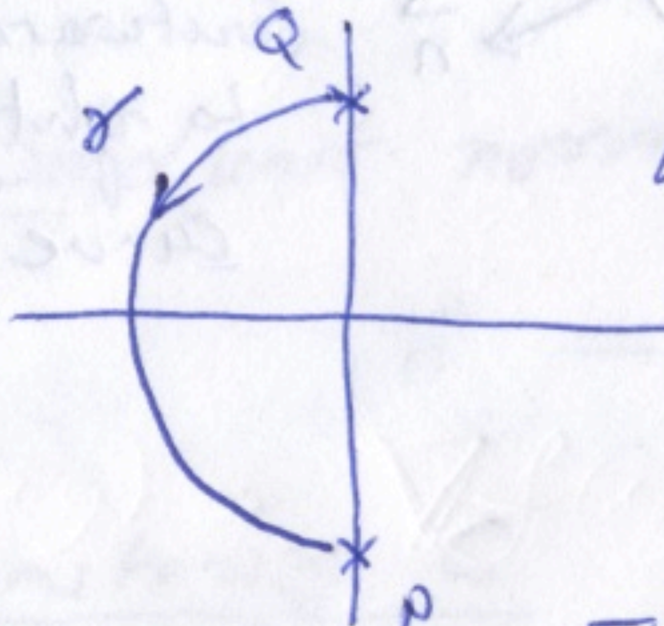
$\vec{r}(t) = tj \quad \begin{cases} x = 0 \\ y = t \end{cases}$

$dx = 0$

$dy = dt$

$M\,dx - N\,dy = 0$

$\Rightarrow$ Flux $= 0$ Makes sense by symmetry

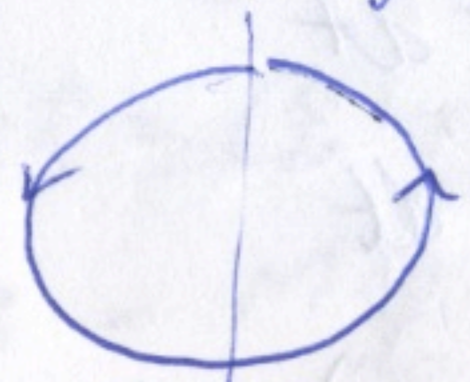


Use same parametrization as before but

$\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

$$\text{Flux} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} dt \cos^2 t = \frac{\pi}{2}$$

Total Flux leaving circle



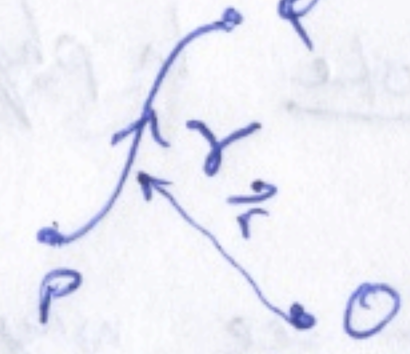
$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Net outward flow

PATH INDEPENDENCE, POTENTIAL FUNCTIONS & CONSERVATIVE FLOWS

The line integral in the previous example was PATH DEPENDENT. There are other examples (such as gravitational potential energy calculated by a work integral) which we expect NOT to depend on the path!

Recall



$$\text{WORK}_{(P,Q,\gamma)} = \int_\gamma \vec{F} \cdot \vec{v}\,dt$$

$$= \int_\gamma \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$$

$$= \int_\gamma \vec{F} \cdot d\vec{r}$$ Parameterization dependent

We say $\vec{F}$ is CONSERVATIVE if

WORK (P, Q, γ) is independent of γ

and often write

$$\text{WORK} = \int_P^Q \vec{F} \cdot d\vec{r} \text{ in that case}$$

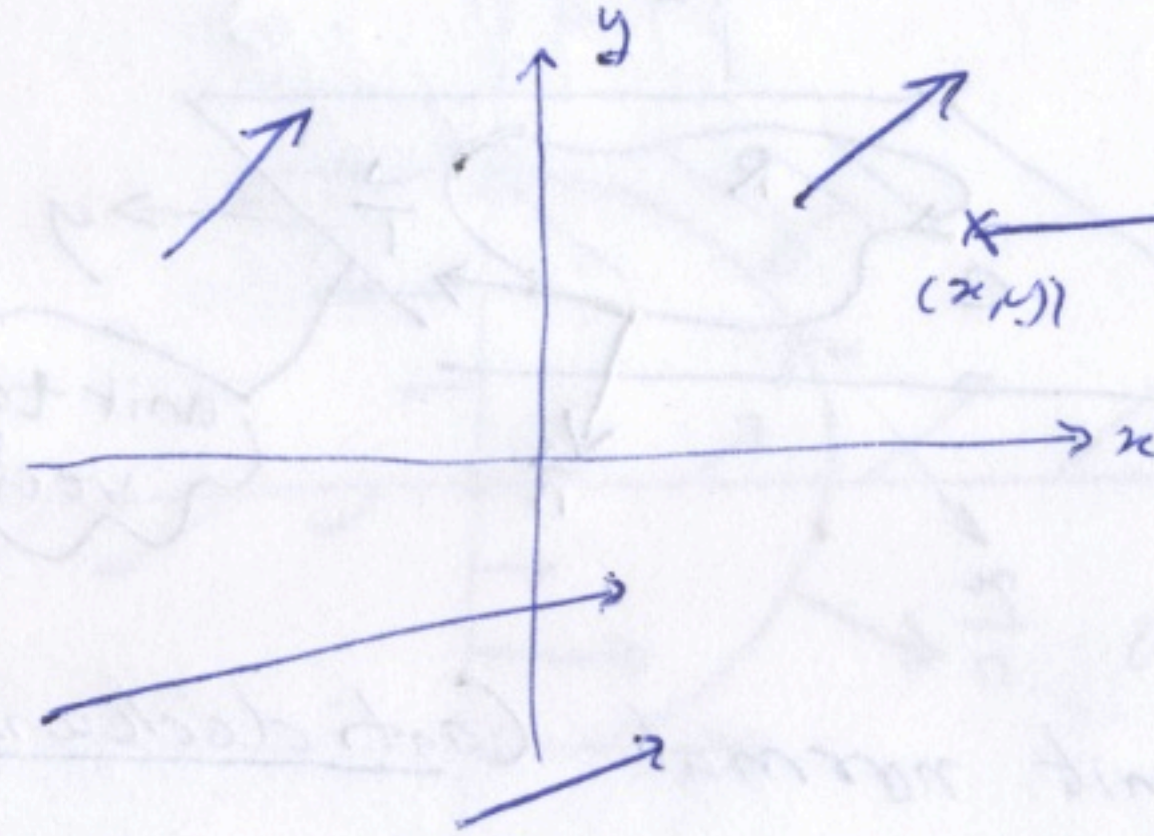
since there is no need to specify γ.

Lecture #16

FLUX (16.2, 16.3)

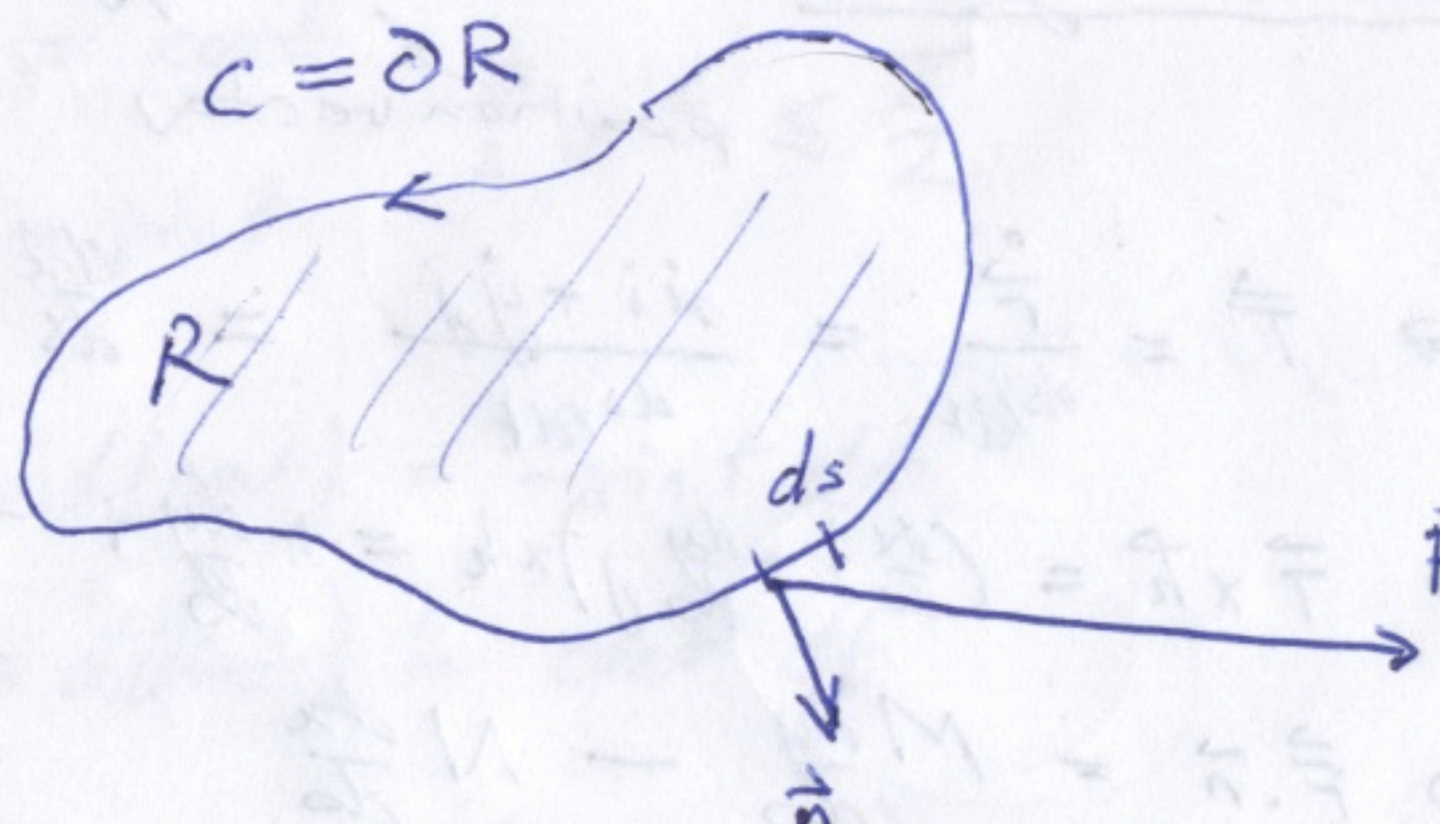
Consider a 2-dimensional velocity vector field

$$\vec{F}(x,y) = M(x,y)\,i + N(x,y)\,j$$



$\vec{F}(x,y)$ measures fluid velocity at $P=(x,y)$

Question Rate of fluid flow from region R



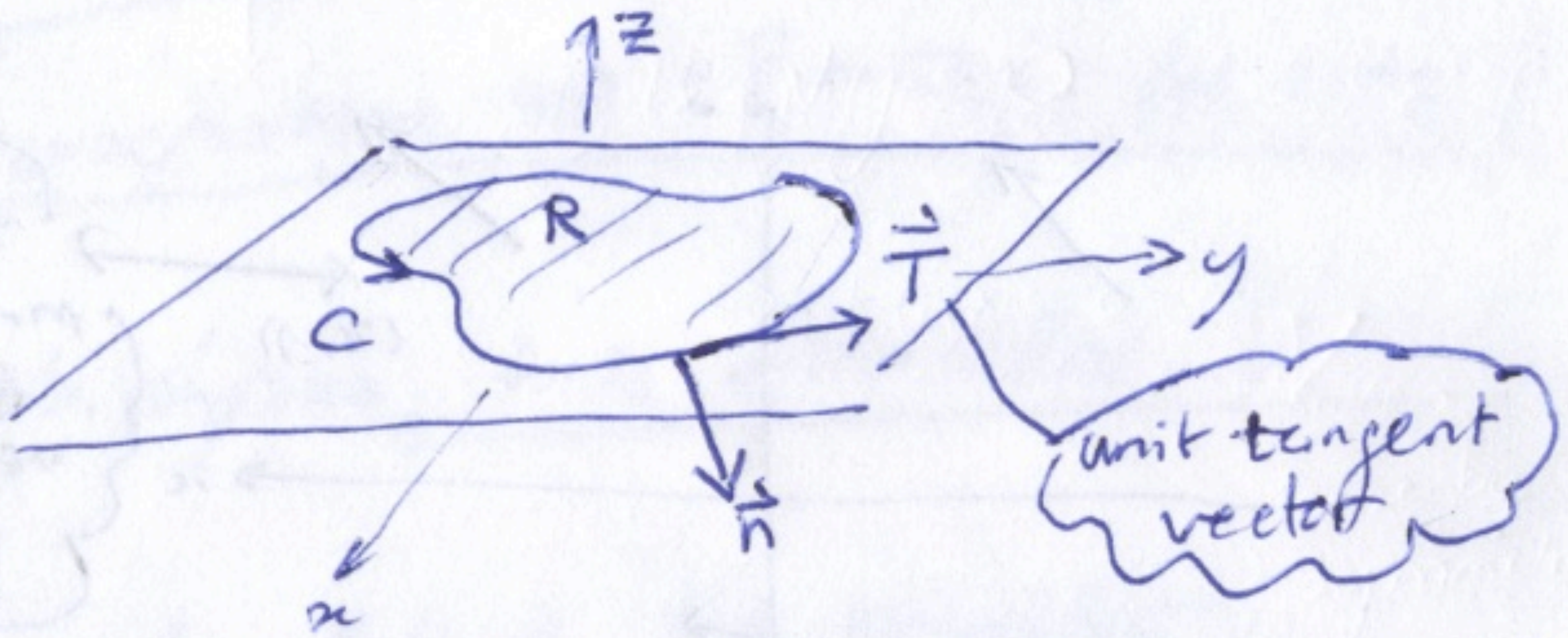
OUTWARD unit normal

Infinitesimal flux $= \vec{F} \cdot \vec{n}\, ds$

Total flux

$$\text{Flux} = \oint_C \vec{F}\cdot\vec{n}\, ds$$ Closed line integral

Finding $\vec{n}$ ⟶ use the 3rd dimension!



Outward unit normal (anticlockwise C)

$$\vec{n} = \vec{T} \times \vec{k} \quad \left[\text{RIGHT HAND RULE}\right]$$

Parameterize C $\quad \vec{r}(t) = xi + yj$ $\quad \begin{cases} x = f(t) \\ y = g(t) \end{cases}$

position vector

$$\Rightarrow \vec{T} = \frac{\dot{\vec{r}}}{ds/dt} = \frac{\dot{x}i + \dot{y}j}{ds/dt} = \frac{dx}{ds}i + \frac{dy}{ds}j$$

$$\Rightarrow \vec{T}\times\vec{k} = \left(\frac{dx}{ds}i + \frac{dy}{ds}j\right)\times k = +\frac{dy}{ds}i - \frac{dx}{ds}j = \vec{n}$$

$$\Rightarrow \vec{F}\cdot\vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$$

$$\boxed{\text{Flux} = \oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C (M\,dy - N\,dx)}$$

[do not need $\vec{n}$ or s]

$$= \int_{t_P}^{t_Q} dt \, \frac{df}{dt} \quad \text{chain rule}$$

$$= f(\vec{r}(t_P)) - f(\vec{r}(t_Q)) = f(P) - f(Q)$$

Depends only on endpoints P & Q!

Look at old example

$$\vec{F} = (x-y)i + yj$$

Try to write $\vec{F} = \nabla f$ for some $f(x,y)$

$$x - y = \frac{\partial f}{\partial x} \quad \text{—— (1)}$$

$$y = \frac{\partial f}{\partial y} \quad \text{—— (2)}$$

(2) says $f = \frac{1}{2}y^2 + g(x)$ — any function $g(x)$

(1) says $x - y = \frac{\partial}{\partial x}\left(\frac{1}{2}y^2 + g(x)\right)$

$$= \frac{\partial g(x)}{\partial x}$$

y-dept

x-dept

↯