On the other hand

$$\frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}} \quad \text{is exact}$$

because

$$d\frac{-1}{\sqrt{x^2+y^2+z^2}} = \frac{x\,dx+y\,dy+z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

<u>Observe</u> Care is required at the origin. In general we must talk about conservative exactness in <u>domains</u>

Lecture #17 PATH INDEPENDENCE, POTENTIAL FUNCTIONS & CONSERVATIVE FLOWS

Recall

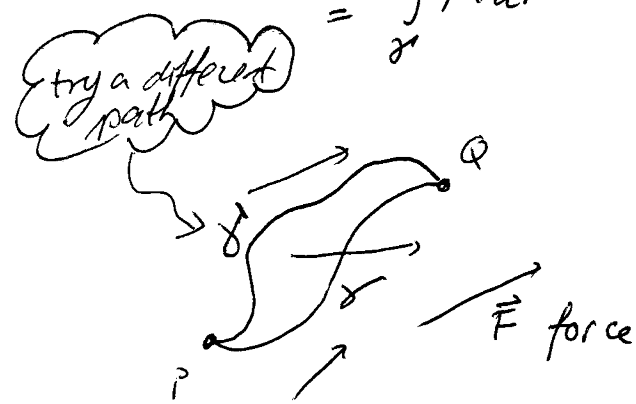
$$\text{WORK}(P,Q,\gamma) = \int_\gamma \vec{F}\cdot\vec{v}\,dt$$

(NB last time discussed flux & dependence on path, today do WORK, these are related questions ...)

$$= \int_\gamma \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$$

$$= \int_\gamma \vec{F}\cdot d\vec{r}$$

Parameterization independent



We call $\vec{F}$ <u>CONSERVATIVE</u> if

WORK(P,Q,γ) is <u>independent of γ</u>

If $\vec{F}$ is conservative we may write

$$\text{WORK} = \int_P^Q \vec{F}\cdot d\vec{r}$$

since there is no need to specify γ.

Another test for conservative fields

If

$$\vec{F} = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = \nabla f$$

we can use

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$$

and similarly for (x, z), (y, z) pairs.

Example

$$\vec{F} = (x-y)\,i + y\,j$$

$$\frac{\partial}{\partial y}(x-y) = -1 \neq \frac{\partial}{\partial x}(y) = 0$$

so $\vec{F}$ is not conservative

Why is this true:

Consider 2-dimensional case –

$$\vec{F}(x,y) = \nabla f(x,y)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) f(x,y)$$

$$= i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y}$$

$$d\vec{r} = d(xi + yj) = i\,dx + j\,dy$$

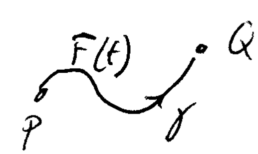
$$\rightarrow \int_P^Q d\vec{r}\cdot\vec{F} = \int_P^Q \left(dx\,\frac{\partial f}{\partial x} + dy\,\frac{\partial f}{\partial y}\right)$$

$$= \int_{t_P}^{t_Q} dt \left[\frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y}\right]$$

choose parameterization $\vec{r}(t)$
$\begin{cases} x = g(t) \\ y = h(t) \end{cases}$ for some path γ b/w P & Q

$$= \int_{t_P}^{t_Q} dt\,\frac{df}{dt} = f(\vec{r}(t_p)) - f(\vec{r}(t_Q))$$

$$= f(Q) - f(P)$$ for ANY choice of γ b/w P & Q



How to decide if $\vec{F}(x,y)$ is conservative:

Example

$$\vec{F} = (x-y)i + yj$$

Try to write $\vec{F} = \nabla f$ for some f:

$$\begin{cases} x - y = \dfrac{\partial f}{\partial x} & \text{——— ①} \\ y = \dfrac{\partial f}{\partial y} & \text{——— ②} \end{cases}$$

② says

$$f = \tfrac{1}{2}y^2 + g(x)$$

ANY FUNCTION $g(x)$

Now ① says

$$x - y = \frac{\partial}{\partial x}\left(\tfrac{1}{2}y^2 + g(x)\right)$$

$$= \frac{dg}{dx}$$

depends on x & y

depends only on x

This $\vec{F}$ is NOT conservative.

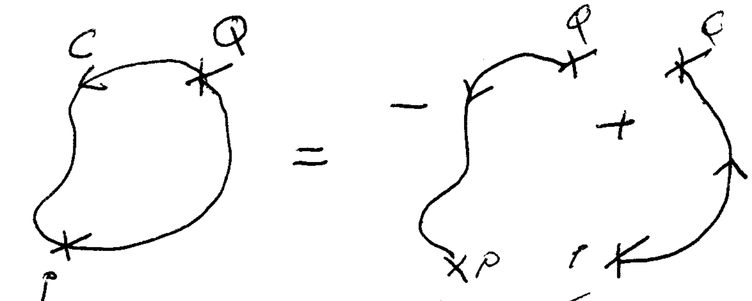
Closed loops & path independence

$$\boxed{\begin{gathered}\oint_C \vec{F}\cdot d\vec{r} = 0 \quad \forall \text{ closed loops } C \\ \Longleftrightarrow \quad \vec{F} \text{ conservative}\end{gathered}}$$

If $\vec{F}$ is conservative then

$\int_P^Q \vec{F}\cdot d\vec{r}$ is conservative

so



But these paths give same integral $\int_P^Q \vec{F}\cdot d\vec{r}$

$$\Rightarrow 0 = \oint \vec{F}\cdot d\vec{r} = -\int^Q \vec{F}\cdot d\vec{r} + \int_P^Q \vec{F}\cdot d\vec{r}$$

$$\Rightarrow \int \vec{F}\cdot d\vec{r} = \int \vec{F}\cdot d\vec{r}$$

POTENTIAL FUNCTION

If $\vec{F} = \nabla f$ we call f a potential function for $\vec{F}$.

For "suitably differentiable" $\vec{F}$

$$\boxed{\vec{F} \text{ conservative} \Leftrightarrow \vec{F} = \nabla f \text{ for some } f}$$

Note, in physics, f is often called V

Example

$$\vec{F} = \frac{-GMm(x i + y j + z k)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \nabla\left(\frac{GMm}{\sqrt{x^2+y^2+z^2}}\right)$$

Because $\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{(-\frac{1}{2})2x}{(x^2+y^2+z^2)^{3/2}}$

Potential $\propto \frac{1}{\text{distance}}$

Differential forms

Consider $\vec{F} = Mi + Nj + Pk$

$$\Rightarrow \vec{F}\cdot d\vec{r} = \underbrace{M\,dx + N\,dy + P\,dz}$$

Often called a differential (1-) form.

When $\vec{F} = \nabla f$, the differential form

$$\boxed{\vec{F}\cdot d\vec{r} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df}$$

is called EXACT.

We can test EXACTNESS the same way we did for conservativeness.

$$(x-y)dx + y\,dy$$

is not exact because

$$\frac{\partial}{\partial y}(x-y) \neq \frac{\partial}{\partial x}y$$