GREEN'S THEOREM IN THE PLANE

Both work integrals & flux integrals dealt with line integrals of the form

$$\int_{\gamma} (M\,dx + N\,dy). \qquad \int_{P}^{Q}{}_{\gamma}$$

In fact, when γ is a closed curve C  we can

(for "sufficiently differentiable" vector fields $\vec{F} = M\hat{i} + N\hat{j}$)

express the line integrals as a double integral over the region R enclosed by C

This is known as GREEN'S THEOREM

Example

$$\vec{F} = (x-y)\hat{i} + x\hat{j} \qquad \begin{cases} M = (x-y) \\ N = x \end{cases}$$

Flux leaving circle $x^2+y^2=1$

Using $\oint_C \vec{F}\cdot\vec{n}\,ds$ we found $FLUX = \pi$

Lets use the divergence form

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}x = 1$$

$$FLUX = \int_R \operatorname{div}\vec{F}\,dA$$

$$= \int_R dA$$

$= \text{Area}$  unit disc

$= \pi$

<u>Notice</u> divergence is opposite to the gradient

gradient : function $\longmapsto$ vector field

divergence : vector field $\longmapsto$ function

<u>Hence</u> total flux

$$\text{FLUX} = \int_R dA \ \operatorname{div} \vec{F}$$

Now we have two expressions for the flux

$$\text{FLUX} = \int_R dA \operatorname{div} \vec{F}$$

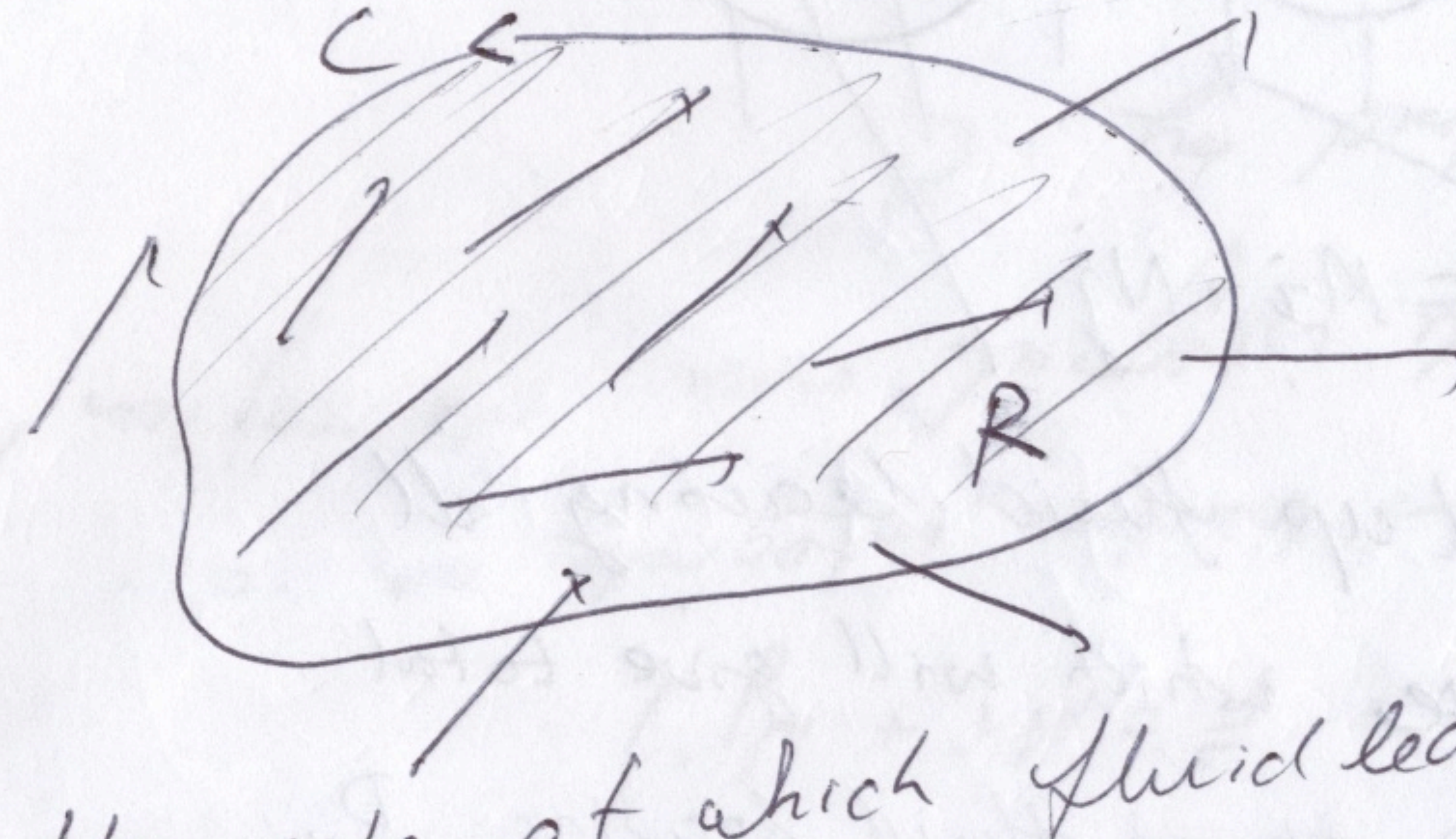
$$\text{FLUX} = \oint_{C=\partial R} \vec{F}\cdot\vec{n}\, ds$$

GREEN'S THEOREM SAYS THEY ARE EQUAL

$$\boxed{\oint_{C=\partial R} \vec{F}\cdot\vec{n}\, ds = \int_R dA \operatorname{div} \vec{F}}$$

Obtaining Green's theorem from <u>FLUXES</u>

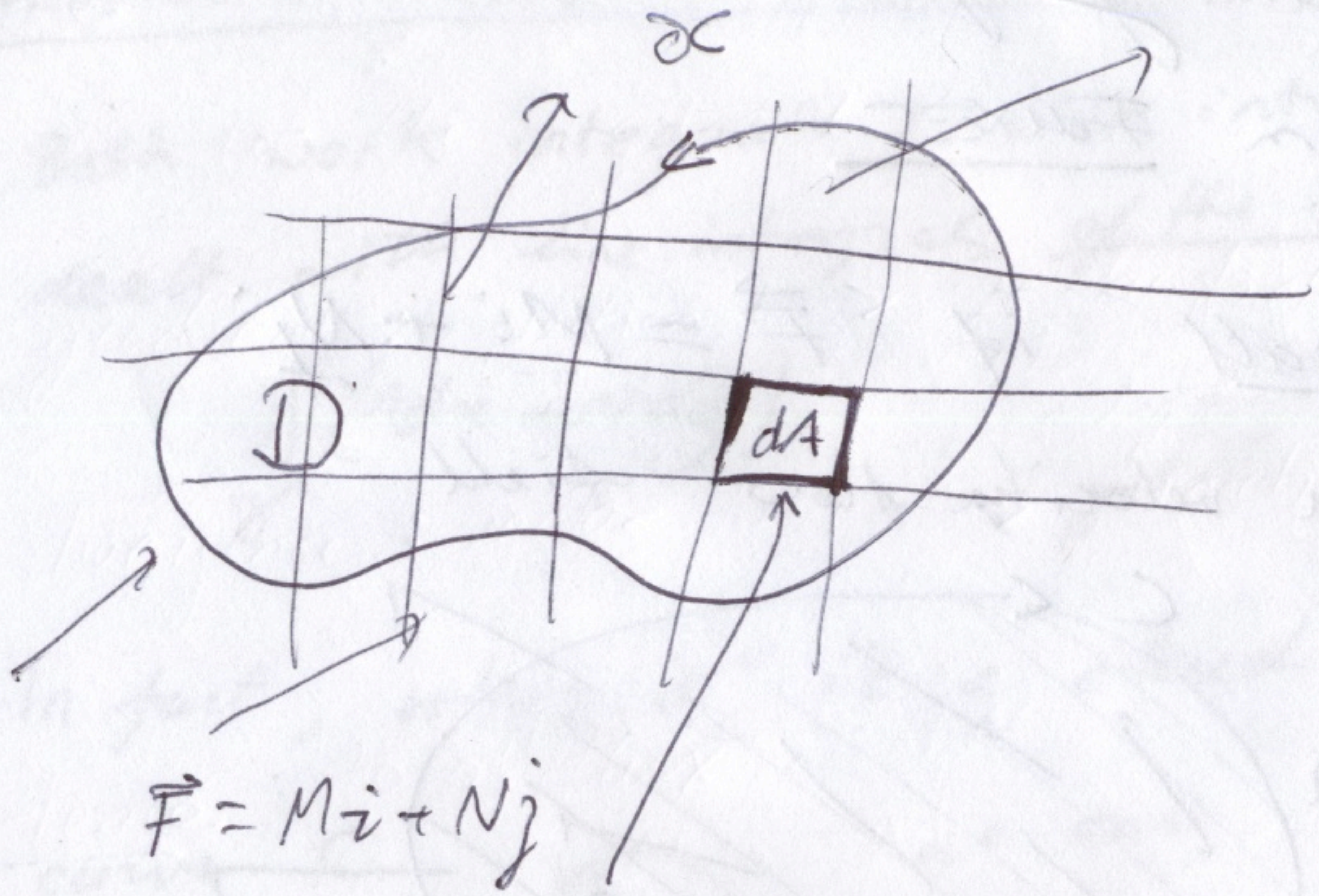
<u>Recall</u> if $\vec{F} = Mi + Nj$ is a velocity flow field



the rate at which fluid leaves the region R is given by the flux

$$\boxed{\text{FLUX} = \oint_C \vec{F}\cdot\vec{n}\, ds = \oint_C (N\,dx - M\,dy)}$$

Now <u>break</u> R into many <u>smaller regions</u>:

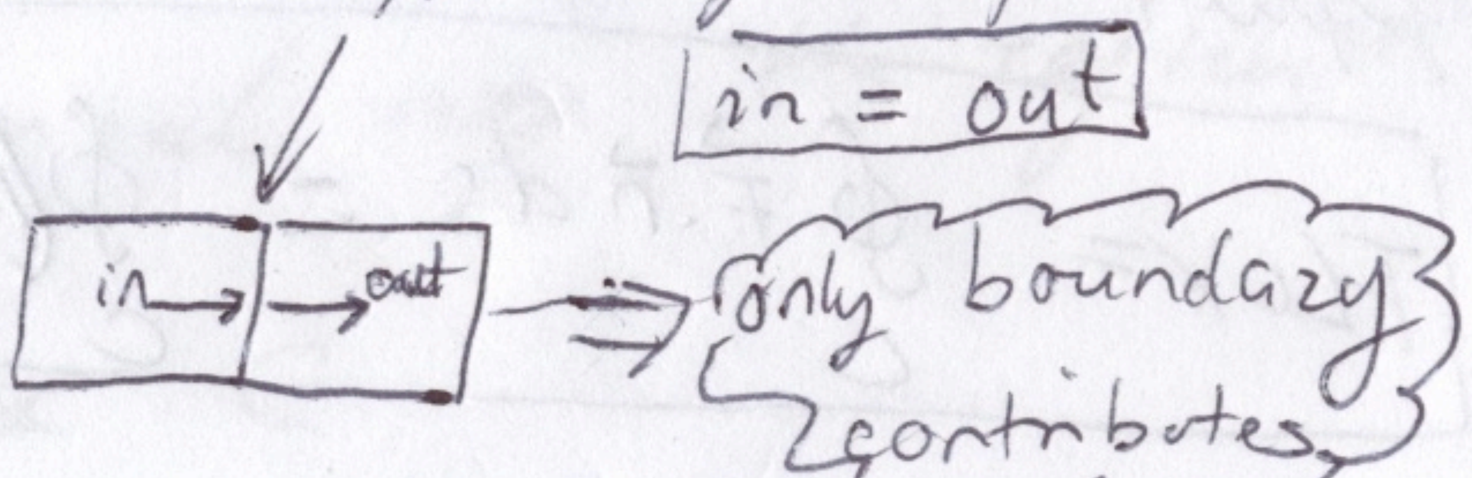


$\vec{F} = Mi + Nj$

Add up fluid leaving all these which will give total amount of fluid leaving D.

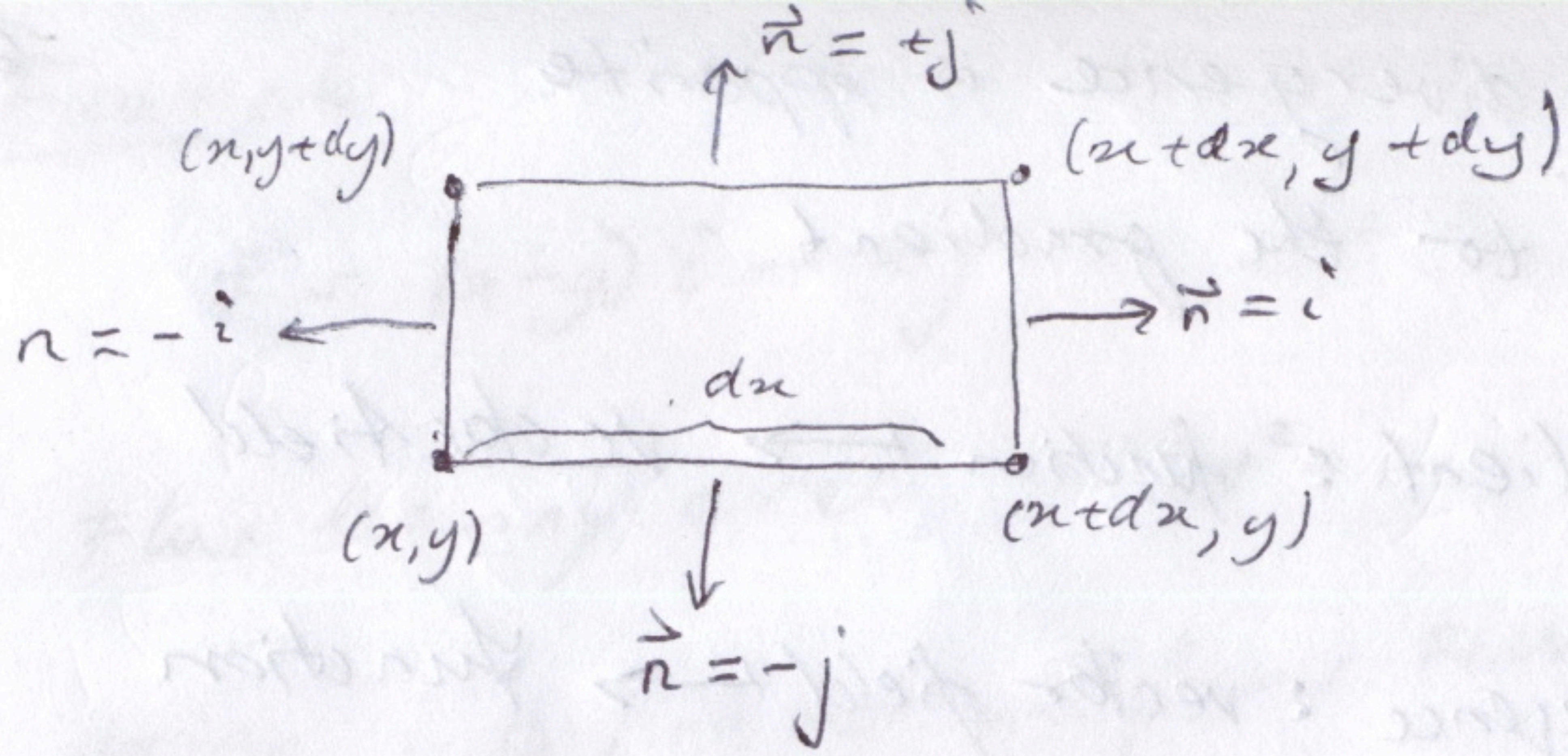
Notice

At every interface $\boxed{in = out}$



Must now compute fluid leaving dA,

i.e. <u>infinitesimal flux</u>



<u>Bottom edge</u> $\vec{F}(x,y) \cdot (-j) \cdot dx = -N(x,y)\,dx$

<u>Top edge</u> $\vec{F}(x,y+dy) \cdot (j)\,dx = +N(x,y+dy)\,dx$

<u>Left edge</u> $\vec{F}(x,y) \cdot (-i)\,dy = -M(x,y)\,dy$

<u>Right edge</u> $\vec{F}(x+dx,y) \cdot i\,dy = M(x+dx,y)\,dy$

<u>Total</u> $dx\left[N(x,y+dy) - N(x,y)\right]$

$+\, dy\left[M(x+dx,y) - M(x,y)\right]$

$\approx dx\,dy\left[\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right] = dA \times$ Flux density

Define

$$\boxed{\operatorname{div}\vec{F} \equiv \vec{\nabla}\cdot\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}$$

divergence / flux density of $\vec{F}$

You can apply Green's theorem to work integrals by remembering

$$\oint \vec{F}\cdot\vec{T}\,ds = \oint_C (M\,dx + N\,dy)$$

which is the same as the flux result when $M \mapsto +N$

$N \mapsto -M$

$$\Rightarrow \boxed{\oint \vec{F}\cdot\vec{T}\,ds = \int_R dA \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]}$$

Lecture #18 EXACT DIFFERENTIAL FORMS

A vector field

$$\vec{F} = Mi + Nj + Pk$$

in $\mathbb{R}^3$ (with its canonical metric) determines a differential form

$$M\,dx + N\,dy + P\,dz$$

If $\vec{F}$ is conservative

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

the we call the differential form

$$M\,dx + N\,dy + P\,dz$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= df(x,y,z)$$

<u>EXACT</u>

You <u>test</u> for exactness of differential forms just as for whether $\vec{F}$ is conservative by looking at higher partial derivatives.