Applying GREEN'S THEOREM TO WORK INTEGRALS

Recall for $\vec{F} = Mi + Nj$

$$\text{WORK} = \oint_C \vec{F}\cdot d\vec{R}$$

$$= \oint_C (M\,dx + N\,dy)$$

$$\text{FLUX} = \oint_C (N\,dx - M\,dy)$$

$$= \int_R dA \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

$$\Rightarrow \text{WORK} = \int_R dA \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

This is sometimes called the "curl form" of GREEN'S THEOREM BECAUSE in 167 we will learn the curl of a three dimensional vector field

$$\nabla \times \vec{F} \equiv \text{curl}\,\vec{F}$$

$$\equiv \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (iM + jN + kP)$$

$$= k\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$+ j\left(-\frac{\partial P}{\partial x} + \frac{\partial M}{\partial y}\right)$$

$$+ i\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)$$

$$\Rightarrow k\cdot(\text{curl}\,\vec{F}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\Rightarrow \boxed{\text{WORK} = \int_R dA\; k\cdot \text{curl}\,\vec{F}}$$

Recall the divergence of $\vec{F} = Mi + Nj$

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial x}$$

Notice

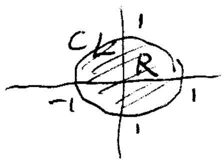
gradient : function $\longmapsto$ vector field

divergence : vector field $\longmapsto$ function

<u>Example</u> $\vec{F} = (x-y)i + xj$

$$\begin{cases} M = (x-y) \\ N = x \end{cases} \quad \text{components}$$

compute flux leaving circle $x^2+y^2 = 1$



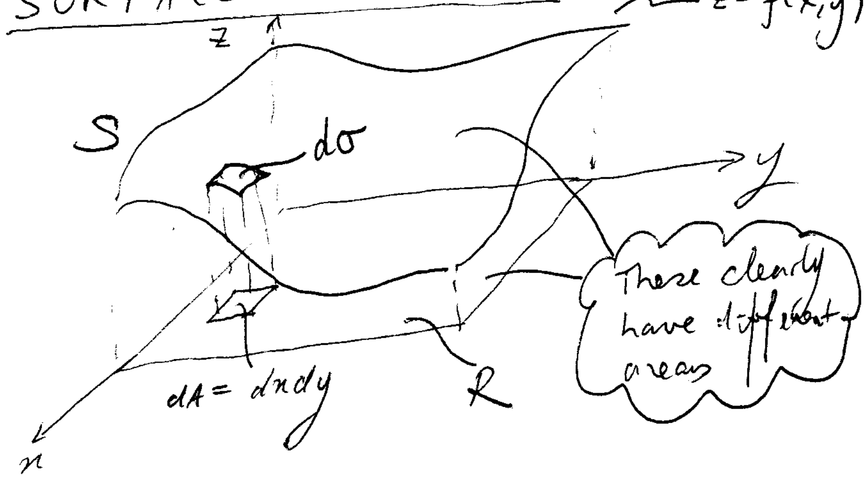
Using $\oint_C \vec{F}\cdot\vec{n}\,ds = \text{FLUX}$ we found

$$\text{FLUX} = \pi$$

using divergence form

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}x = 1$$

<u>SURFACE INTEGRALS</u>



How to compute the area of curved surfaces and even integrate functions over surfaces.

Could we relate $d\sigma$ to dA ??

$$d\sigma = \rho(x,y)\,dA$$

If so could express

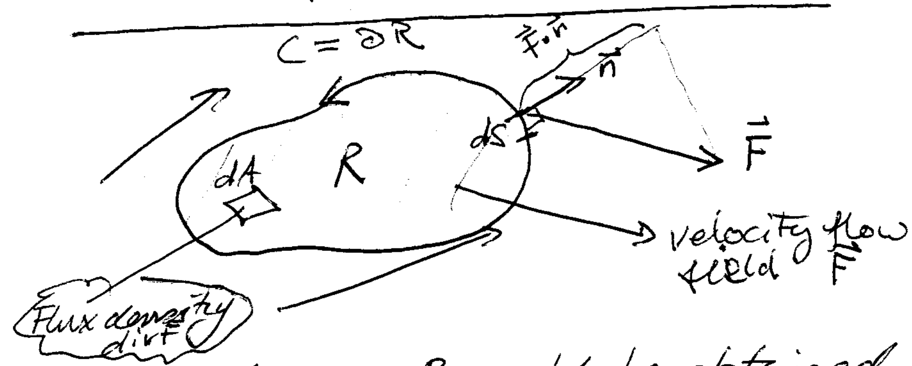
$$\text{SURFACE AREA} = \int_S d\sigma = \iint_R \rho(x,y)\,dA$$

ordinary area integral

$\boxed{\text{READ ch 16.5}}$

Lecture #19

GREENS THEOREM FOR FLUX INTEGRALS ch 16.4



Flux leaving R could be obtained two ways

1. Add up infinitesimal flux leaving boundary

   $\vec{F}\cdot\vec{n}\, ds \rightarrow$ line integral

2. Add up infinitesimal flux leaving area dA

   $$\underbrace{\text{div}\,\vec{F}}_{\text{flux density}}\, dA$$

Equating these gave Green's theorem

$$\boxed{\oint_{C=\partial R} \vec{F}\cdot\vec{n}\, ds = \int_R dA\, \text{div}\,\vec{F}}$$

FIND THE WORK TO TRAVERSE A ~~A~~CLOCKWISE CIRCLE $x^2+y^2=1$ IN THE FORCE FIELD

$$\vec{F} = -yi + xj \qquad M = -y \quad N = x \qquad \text{HURRICANE}$$

$$k \cdot \operatorname{curl} \vec{F} = \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)$$

$$= 2$$

$$\text{WORK} = \int_R dA \; k \cdot \operatorname{curl} \vec{F}$$

$$= 2 \int_R dA = 2\pi \leftarrow$$

<u>Positive</u> $\Rightarrow$ GAIN <u>ENERGY</u>

(work done by the wind!)

$$\text{FLUX} = \int_R \operatorname{div} \vec{F} \, dA$$

$$= \int_R 1 \, dA$$

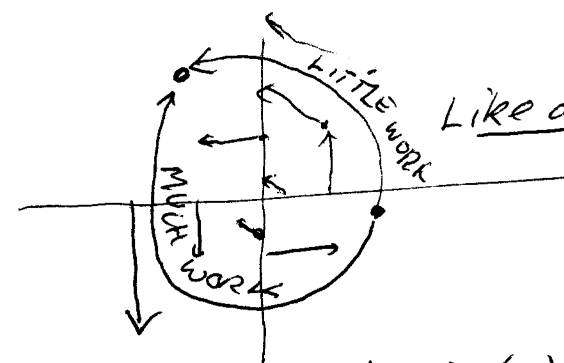
$$= \text{Area (unit disc)}$$

$$= \pi$$

<u>Notice</u> If $\operatorname{div} \vec{F} = 0$ the flux leaving a region vanishes b/c

$$\int_R dA \operatorname{div} \vec{F} = 0$$

<u>Example</u> $\vec{F} = -yi + xj$



<u>Like a Hurricane</u>

Notice $\frac{\partial(-y)}{\partial y} = -1 \neq \frac{\partial}{\partial x}(x) = +1$

NOT CONSERVATIVE

But no flux leaves any region.

$\Rightarrow$ PATH INDEPENDENCE OF FLUX