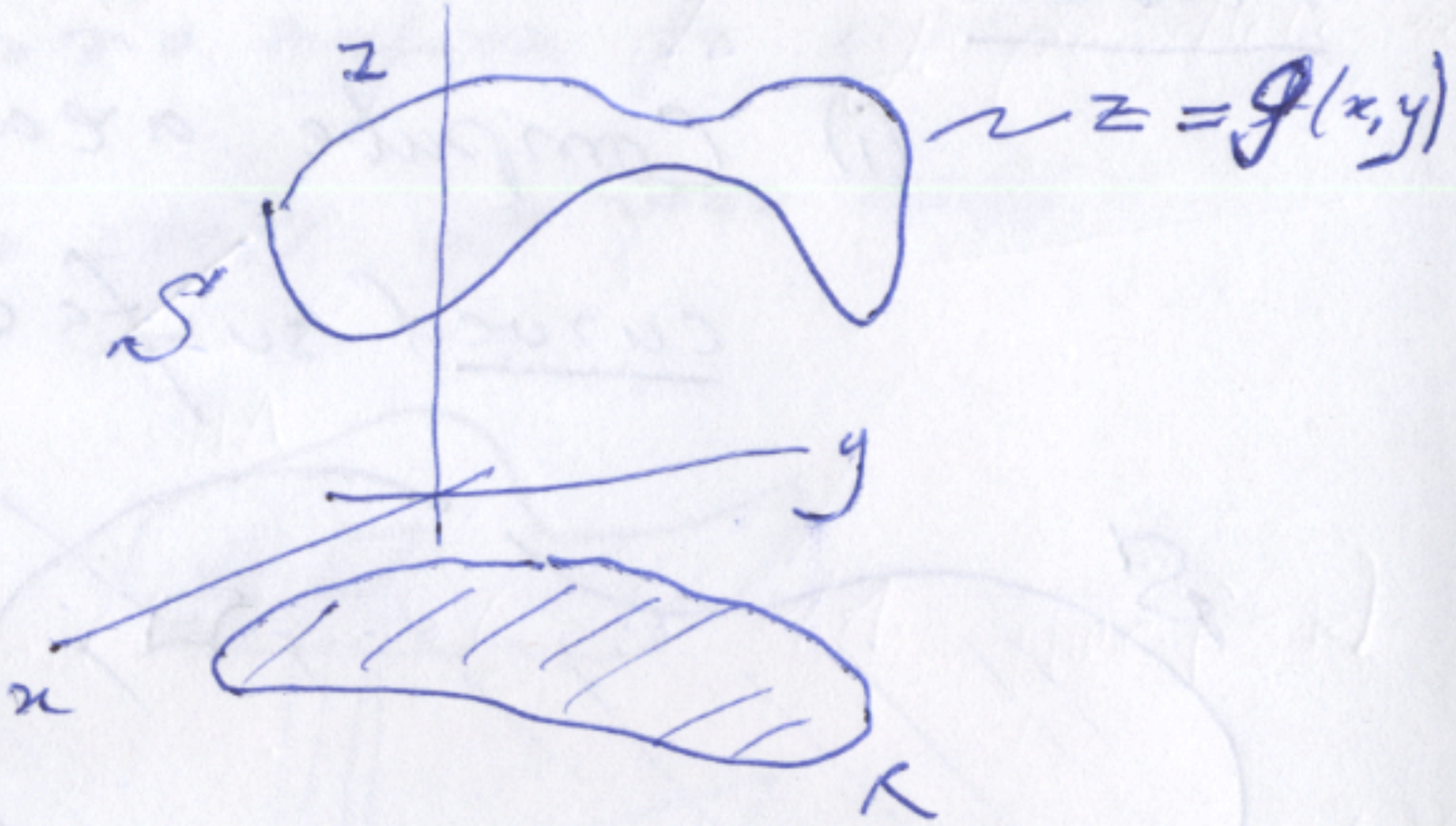
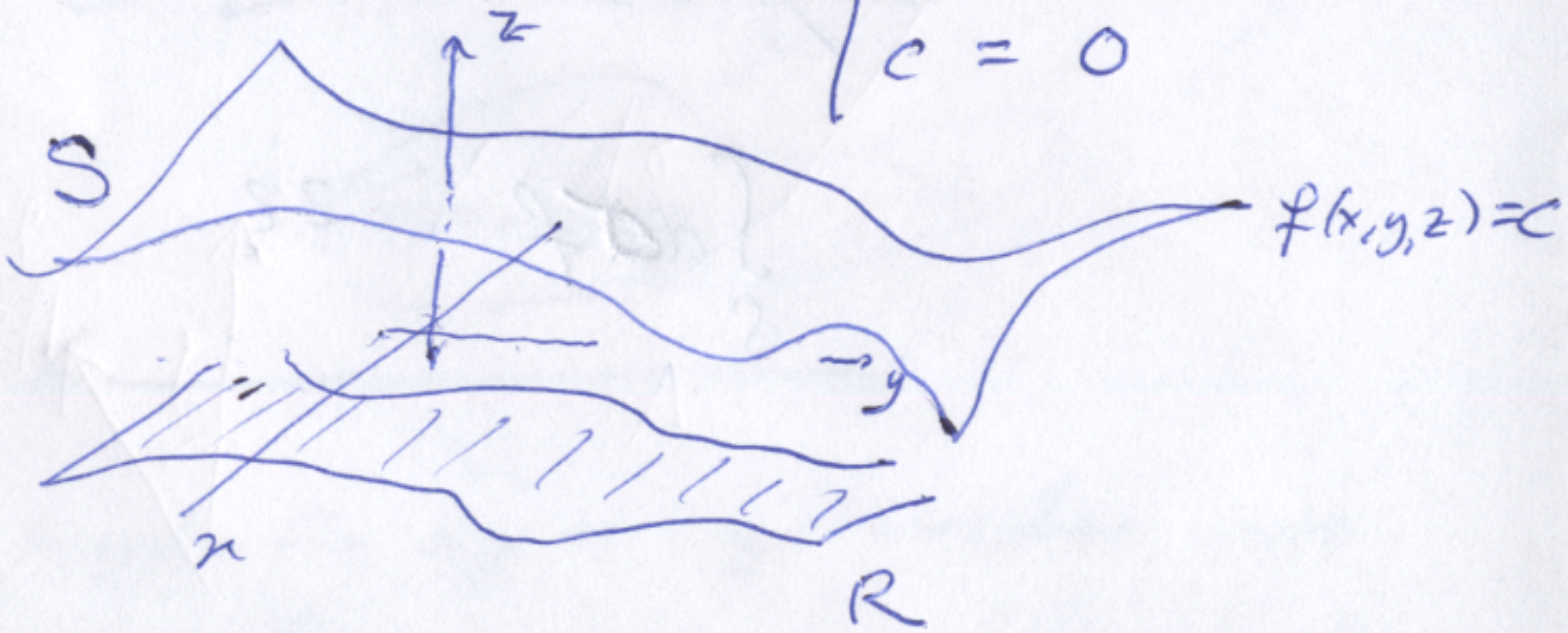
Suppose S is given as a function over a region R in xy-plane



More generally, we can try to describe S by a single relation

$$f(x,y,z) = c \quad \underline{\text{constant}}$$

The above example is $\begin{cases} f = z - g(x,y) \\ c = 0 \end{cases}$



$$\Rightarrow |\cos\gamma| = \frac{\sqrt{1-(x^2+y)^2}}{1}$$

$$\Rightarrow \text{SURFACE AREA} = \int_R \frac{dA}{\sqrt{1-x^2-y^2}}$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,dr \frac{1}{\sqrt{1-r^2}}$$

use polar coordinates for x,y plane

$$= 2\pi \int_{r^2=0}^{r^2=1} \frac{\frac{1}{2}\,d(r^2)}{\sqrt{1-r^2}}$$

$$= 2\pi \cdot \frac{1}{2} \left[-\cancel{2}\sqrt{1-r^2} \right]_{r^2=0}^{r^2=1}$$

$$= 2\pi \cdot \cancel{2} \cdot \cancel{\tfrac{1}{2}} \left[-0 + 1 \right]$$

$$= 2\pi$$

The whole sphere would have area 4π.

This is an example of the famous result

$$\boxed{\text{Surface Area of Sphere Radius } R = 4\pi R^2}$$

Find the surface area of the hemisphere $x^2+y^2+z^2=1 \quad z \geq 0$

Here $f(x,y,z) = x^2+y^2+z^2$

$$\nabla f = 2xi + 2yj + 2zk$$

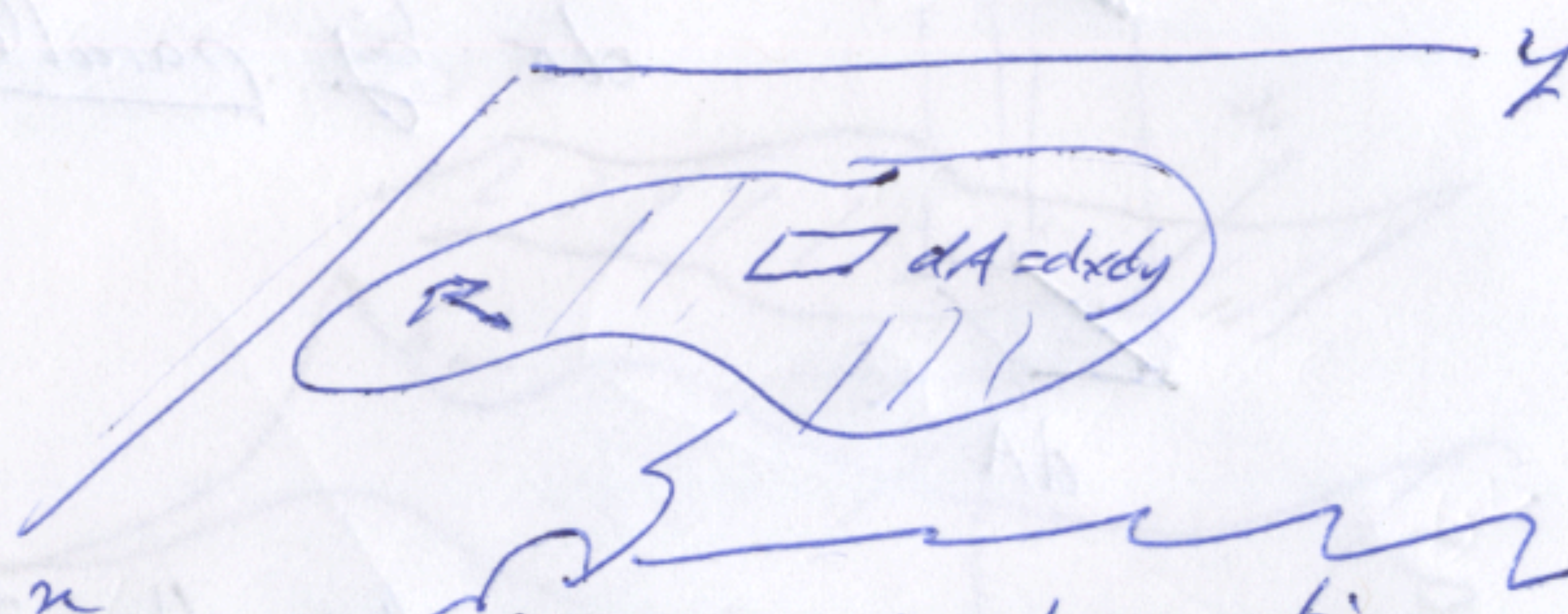
$$\frac{\nabla f}{|\nabla f|} = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|\vec{r}|}$$

$$\vec{p} = k$$

$$\left|\frac{\nabla f}{|\nabla f|}\cdot k\right| = \frac{|z|}{\sqrt{x^2+y^2+z^2}} = |\cos\gamma|$$

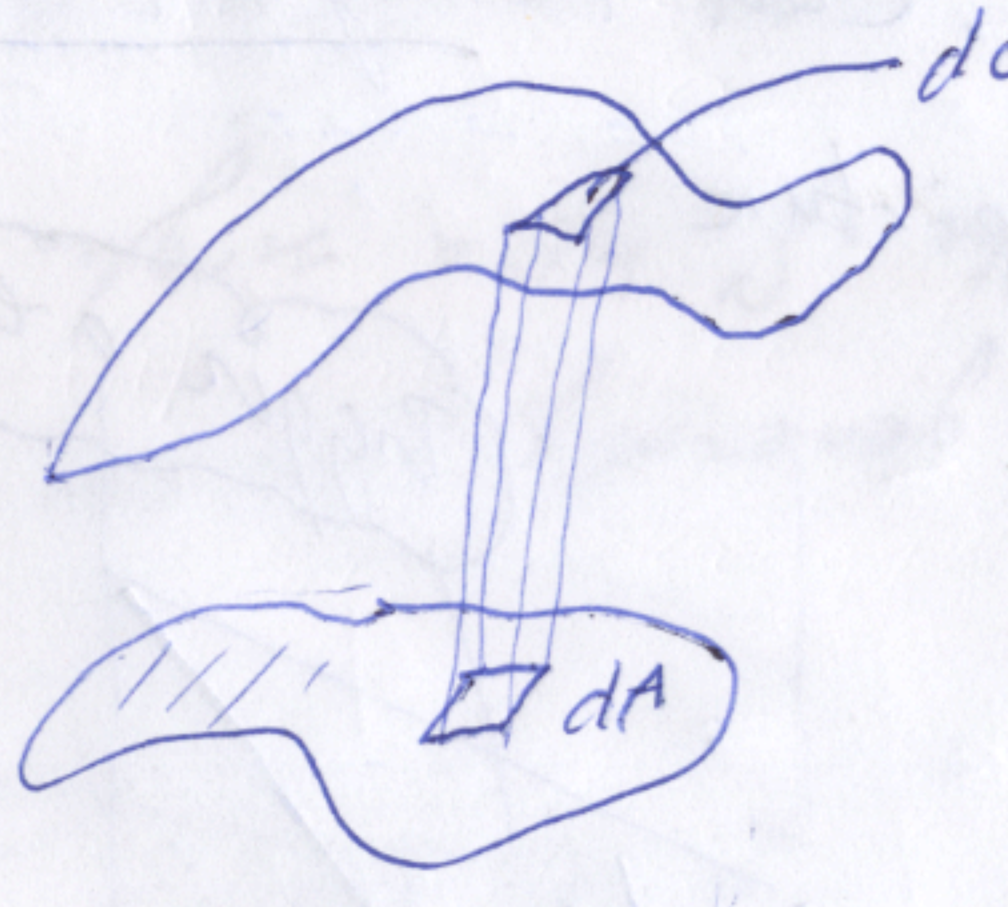
Now $z^2+y^2+z^2=1$ determines z

We already know how to integrate over regions R in the xy-plane



bounds of integration give iterated integral

Try to use this to perform surface integral:



Above each dA lives an infinitesimal piece of surface area $d\sigma$

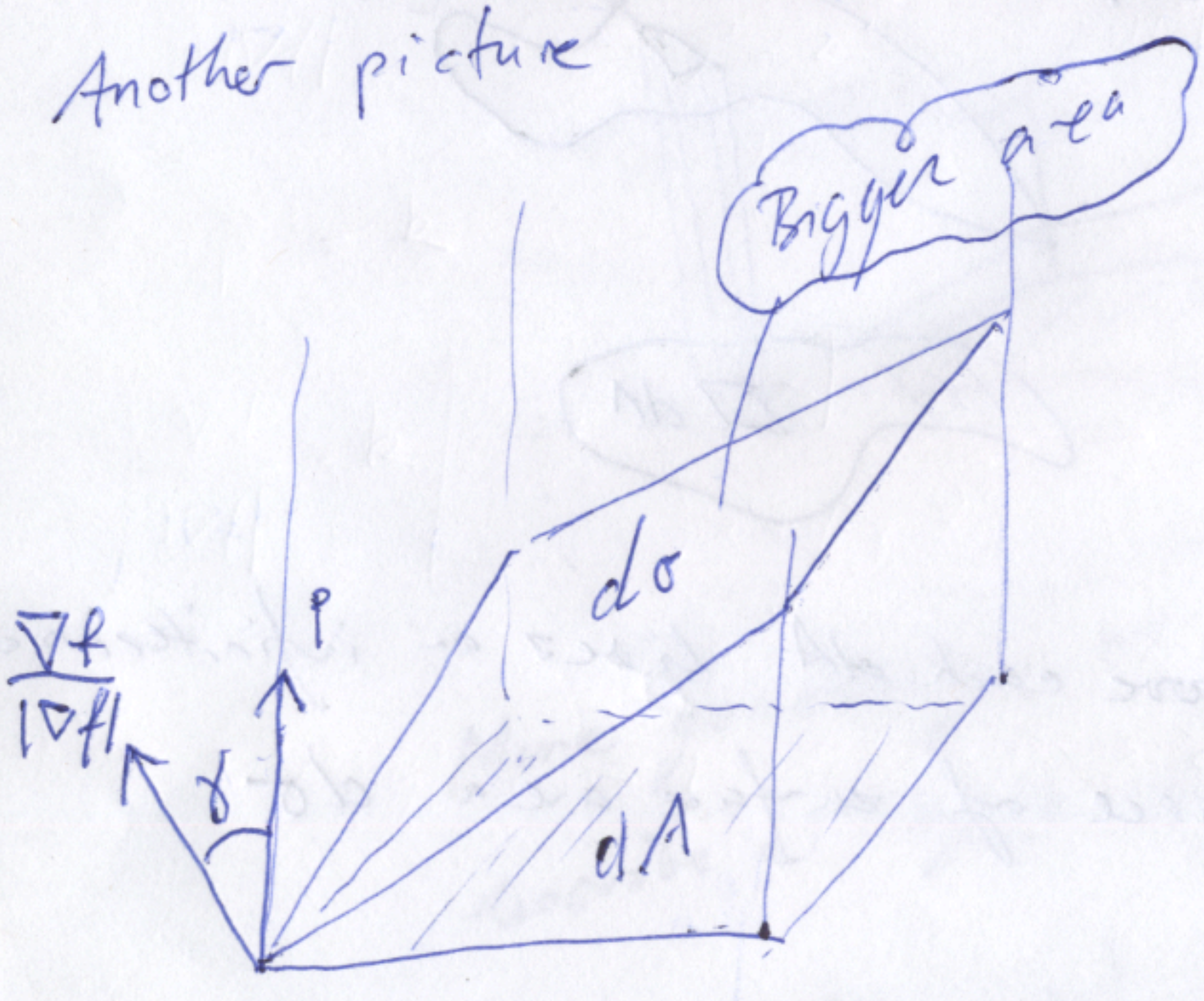
Calculate $d\sigma$:

Approximate $d\sigma$ by parallelogram

Call $\vec{p}$ the unit normal to dA (here $\vec{p} = k$)

Notice that $\dfrac{\nabla f}{|\nabla f|}$ is the unit normal to $d\sigma$. Call the angle b/w these γ

Another picture

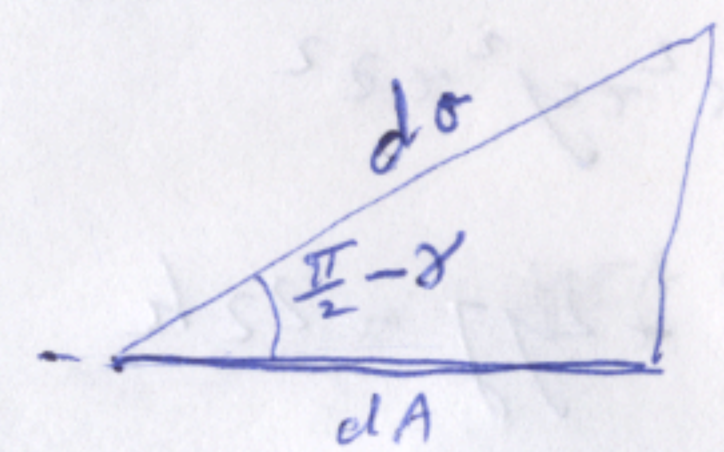


Notice ; when $\gamma = 0$ ~~$|d\sigma| = |dA|$~~

$d\sigma$ and dA are the same size

when $\gamma \to \frac{\pi}{2}$ $\quad \dfrac{d\sigma}{dA} \longrightarrow \infty$

Notice In profile we see



$\Rightarrow$ suspect

$d\sigma \sin(\frac{\pi}{2} - \gamma) = dA$

ie $d\sigma |\cos\gamma| = dA$

Infact this is true : The book has a clunky proof.

Hence

$$\text{Surface area} = \int_R \frac{dA}{|\cos\gamma|} = \int_R dA \frac{1}{\left|\frac{\nabla f}{|\nabla f|} \cdot \vec{p}\right|}$$

$$= \int_R \frac{|\nabla f|\, dA}{|\nabla f \cdot \vec{p}|}$$

Lecture #20 — SURFACE INTEGRALS 16.5

Problem

(i) Compute area of a <u>curved</u> surface

$$\text{AREA} = \int_S d\sigma$$

(ii) Integrate functions (or vector fields) over surfaces

$$\int_S d\sigma\, f \quad ??$$