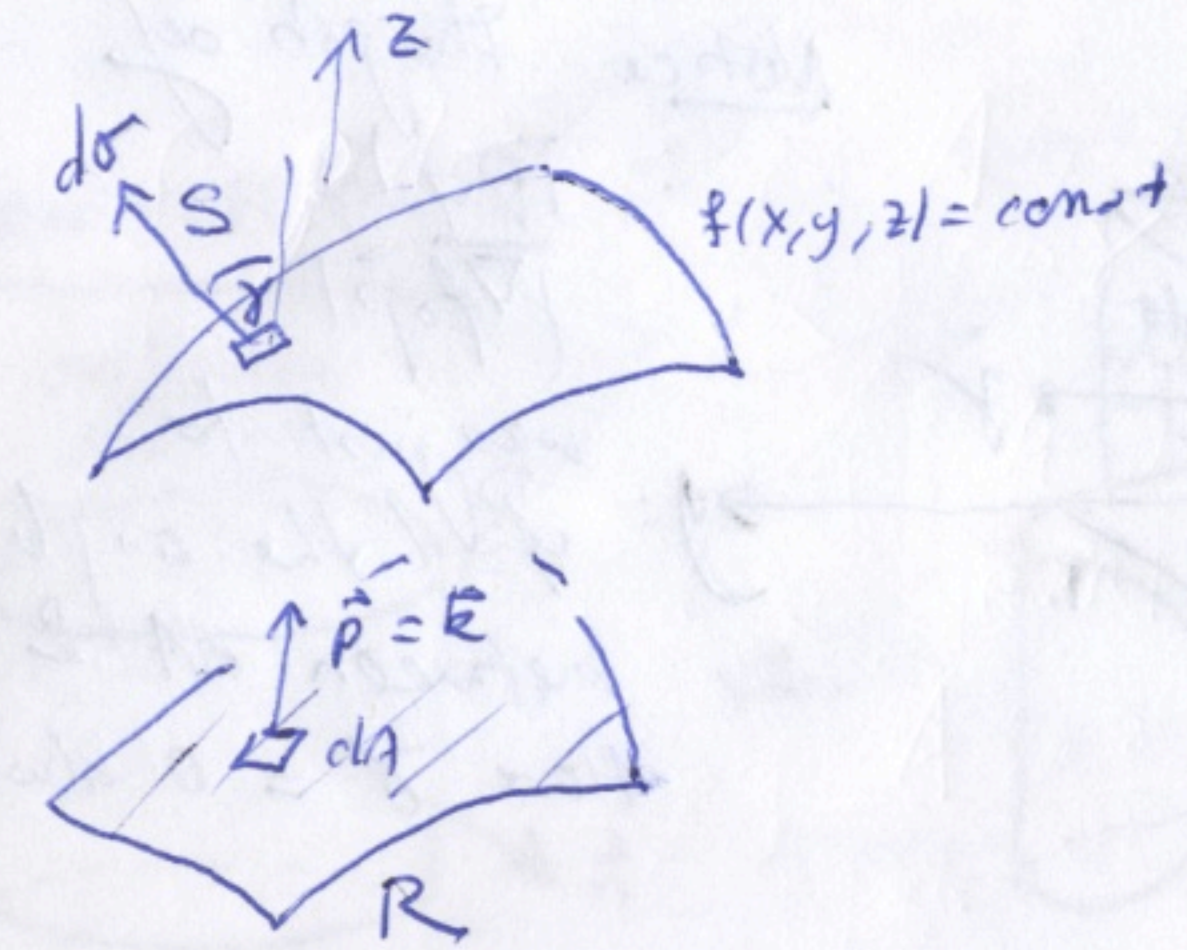
Lecture # 21 SURFACE INTEGRALS

$$\text{SFCE AREA} = \int_R \frac{dA}{|\cos\gamma|} = \int_R \frac{|\nabla f|\,dA}{|\nabla f\cdot\vec{p}\,|} = \int_S d\sigma$$



Surface integrals can be defined similarly. Suppose the function $g(x,y,z)$ is defined for all $p = (x,y,z)$ in S then

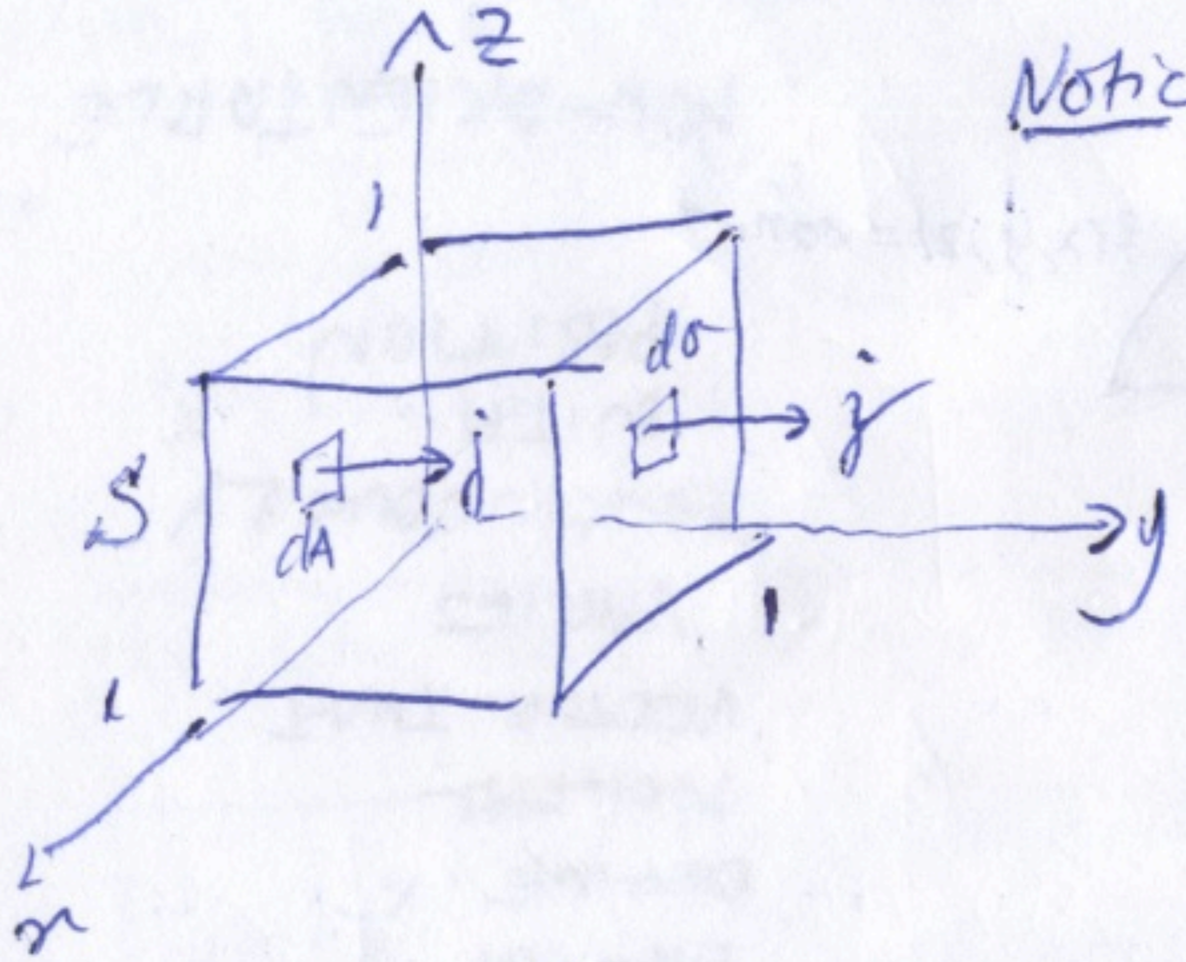
$$\boxed{\int_S g\,d\sigma = \int_R \frac{g(x,y,z)\,dA\,|\nabla f|}{|\nabla f\cdot\vec{p}\,|}}$$

Notice that LHS doesn't depend on f

RHS seems to depend on $f(x,y,z)$, in fact, just like parametrization, any $f(x,y,z)$ determining S is ok

Example

Compute the average value of the function $g(x,y,z) = xyz$ over the surface S of the cube bounding the volume $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$



Notice the job of

$$\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}$$

was just to find the angle between dA & $d\sigma$

Here $\gamma = 0$ always

$$\langle g \rangle = \frac{\int_S g\, d\sigma}{\int_S d\sigma} = \frac{1}{6} \int_S g\, d\sigma$$

six faces with unit area

$$= \frac{1}{6} \Big[\int_0^1 dx \int_0^1 dy\, xyz \Big|_{z=0} + \int_0^1 dx \int_0^1 dy\, xy \Big|_{z=1}$$

$$+ \int_0^1 dx \int_0^1 dz\, xyz \Big|_{y=0} + \int_0^1 dx \int_0^1 dz\, xyz \Big|_{y=1}$$

$$+ \int_0^1 dy \int_0^1 dz\, xyz \Big|_{x=0} + \int_0^1 dy \int_0^1 dz\, xyz \Big|_{x=1} \Big]$$

By symmetry the centre of mass lies on the z-axis.

First moment in z-direction

$$\int_S z\rho\, d\sigma = \rho \int_R \frac{dA\, z}{\cos\gamma}$$

$$= \rho \int_R \frac{dA\, z}{|k \cdot \hat{n}|}$$

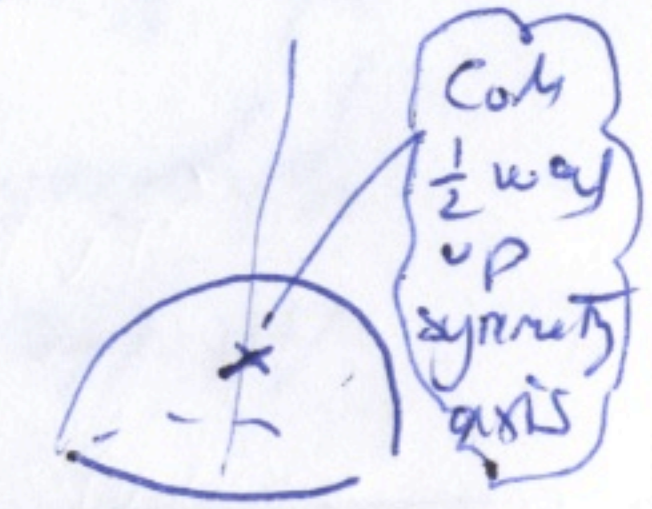
$$= \rho \int \frac{dA\, z}{\left| k \cdot \frac{x i + y j + z k}{r} \right|}$$

$$= \rho \int \frac{dA\, \cancel{z}\, r}{\cancel{z}}$$

$$= \rho r \; 4\pi r^2$$

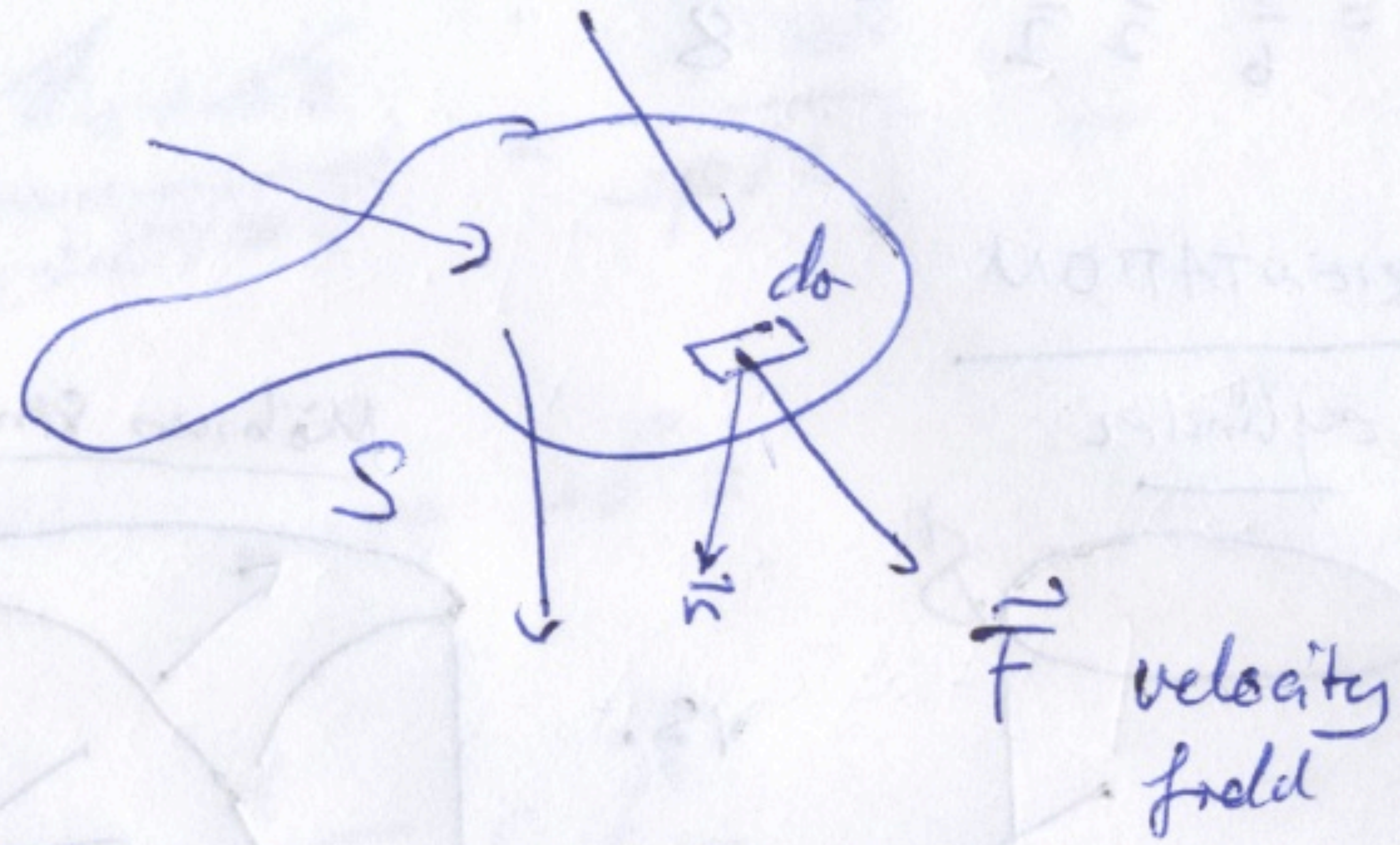
$$\Rightarrow \frac{\int z\rho\, d\sigma}{\text{MASS}} = \frac{\cancel{\rho}\, 4\pi r^3}{2\pi r^2 \cancel{\rho}}$$

$$= \frac{r}{2}$$



FLUX

RECALL, TO COMPUTE FLUID FLOWING ACROSS A SURFACE S



WE NEED TO COMPUTE

$$\boxed{FLUX = \int_S \vec{F}\cdot\vec{n}\, d\sigma}$$

<u>we can now handle such integrals</u>

You might even wonder for closed surfaces $S = \partial V$ bounding a volume V

whether
$$\boxed{\oint_S \vec{F}\cdot\vec{n}\, d\sigma = \int_V dV \operatorname{div}\vec{F}}$$

Actually this is true – ch16.8 & the <u>divergence theorem</u>

<u>EX</u> Calculate the flux of

$$\vec{F} = xi + yj + zk$$

leaving the unit sphere about the origin

<u>Notice</u> $\vec{F} = \vec{n}$ on S

Therefore $\vec{F}\cdot\vec{n} = 1$

$$FLUX = \int_S d\sigma\, \vec{F}\cdot\vec{n} = \int_S d\sigma\, 1 = 4\pi$$

Just the surface area of the sphere.

<u>NOTE</u> $\operatorname{div}\vec{F} = 3$ so if a divergence theorem held

$$4\pi = \int_{S=\partial V} d\sigma\, \vec{F}\cdot\vec{n} = \int_V dV \operatorname{div}\vec{F}$$

$$= 3\int_V dV = 3(\text{volume unit sphere})$$

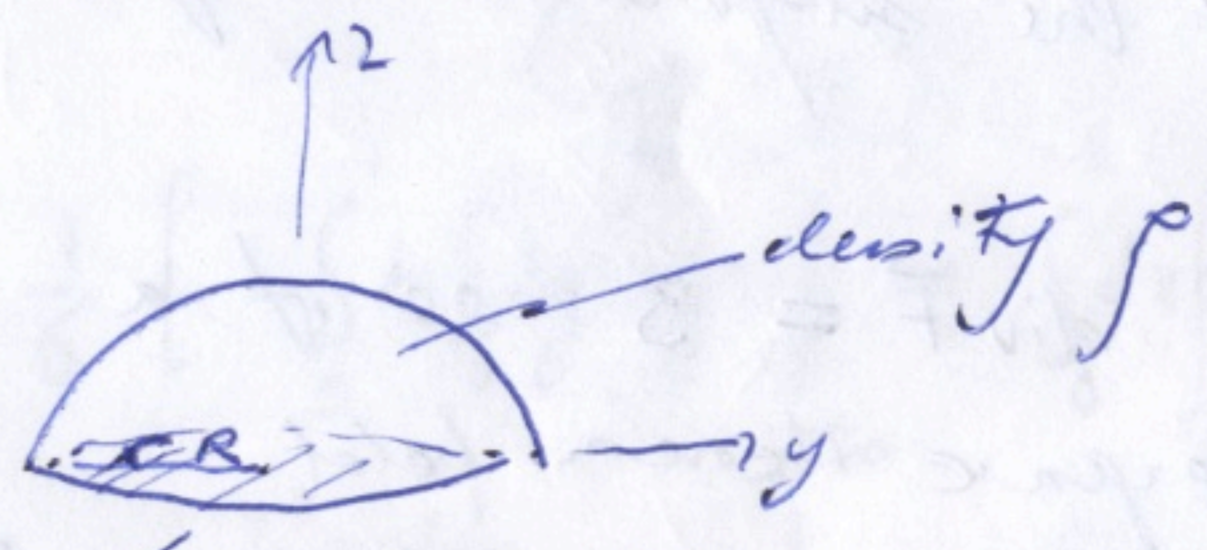
$\Rightarrow$ VOLUME OF UNIT SPHERE $= \frac{4\pi}{3}$

THIN SHELLS

All the area integrals of physical interest we computed for thin lamina can be applied to surfaces. [In fact area integrals are just special cases of surface integrals when the surface happens to be flat!]

Example

Find the centre of mass of a constant density thin hemispherical shell.



surface area $2\pi R^2$

Mass $2\pi R^2 \rho$

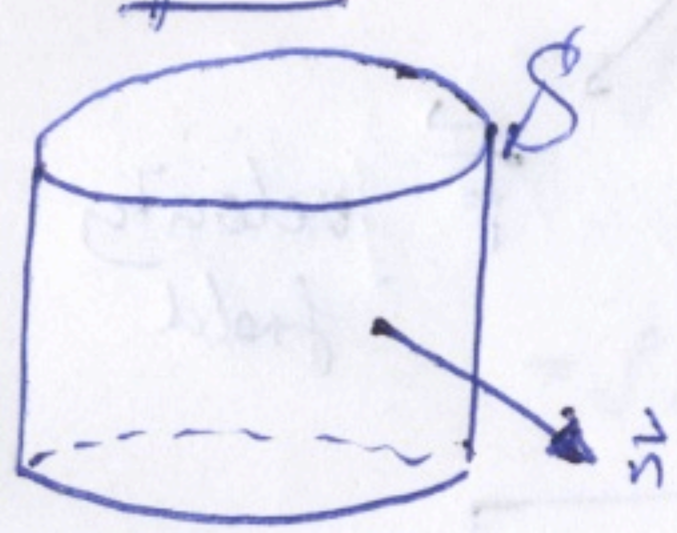
$$= \frac{3}{6}\left[0 + \int_0^1 x\,dx \int_0^1 y\,dy\right]$$

by symmetry

$$= \frac{3}{6}\,\frac{1}{2}\,\frac{1}{2} = \frac{1}{8}$$

ORIENTATION

cylinder vs. Möbius strip



CAN DEFINE AN ~~NORMAL~~ VECTOR THAT VARIES CONTINUOUSLY WITH POSITION	CANNOT DEFINE A NORMAL VECTOR THAT VARIES CONTINUOUSLY WITH POSITION
"ORIENTABLE"	"NON-ORIENTABLE"
"TWO SIDED"	

CLOSED SURFACES HAVE INSIDE & OUTSIDE IN THREE DIMENSIONS! FAILS IN FOUR DIMENSIONS!