$$\text{AREA} = \int_S d\sigma = \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \, \sin\varphi$$

$$= 2\pi \underbrace{\left(-\cos\varphi\right)_0^{\pi}}_{2}$$

$$= 4\pi$$

Lecture #22

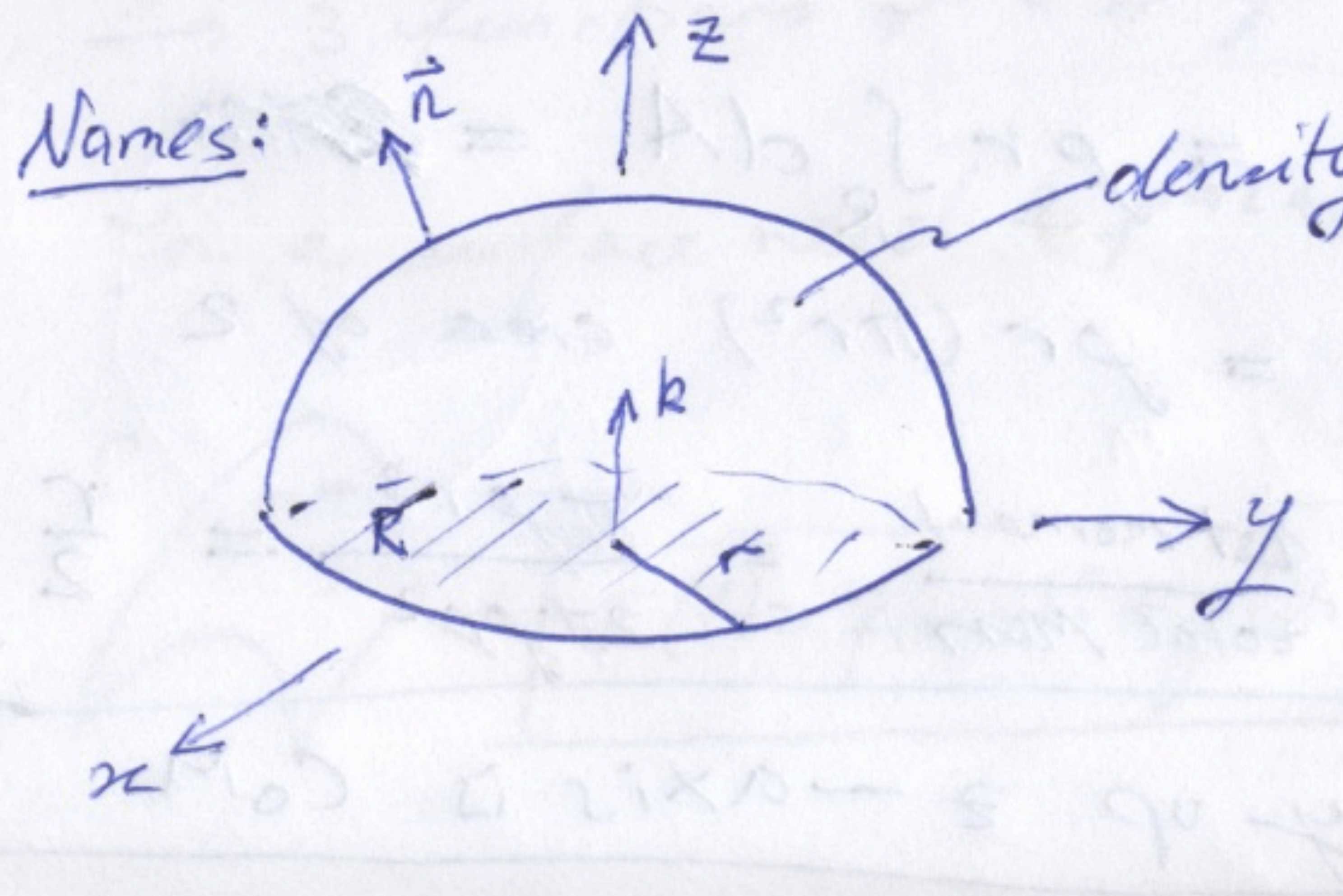
One more surface area/integral example:

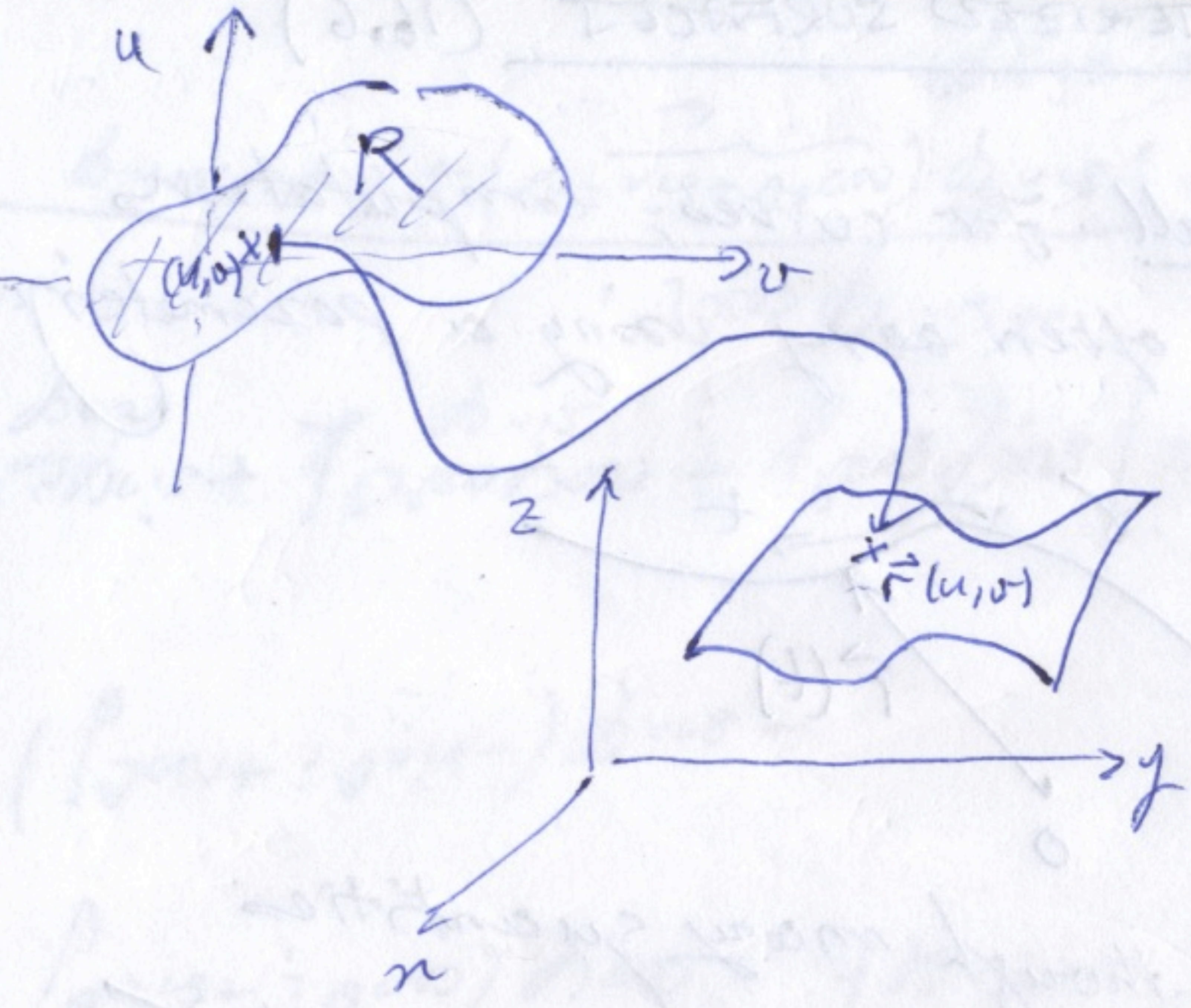
THIN SHELLS

Remark: All the area integrals of physical interest we computed for thin lamina [moments, centre of mass, moments of inertia etc...] apply to surfaces. In fact they are really SPECIAL CASES of surface integrals.

Example Find the centre of mass of a uniform density, thin hemispherical shell.

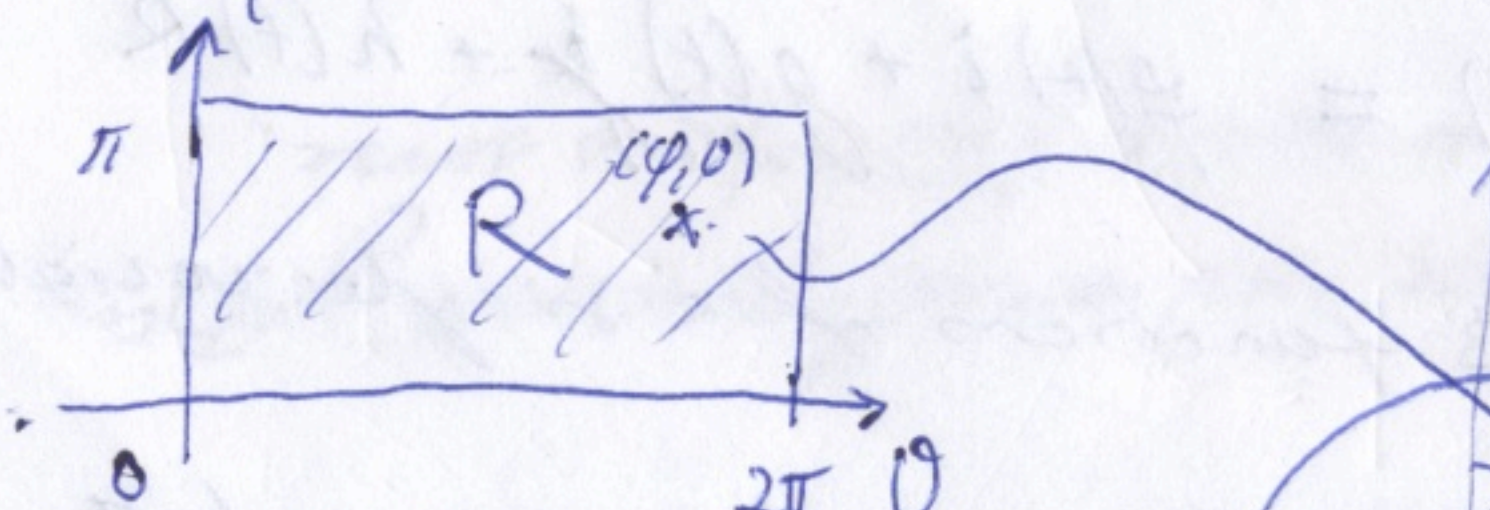
Names:

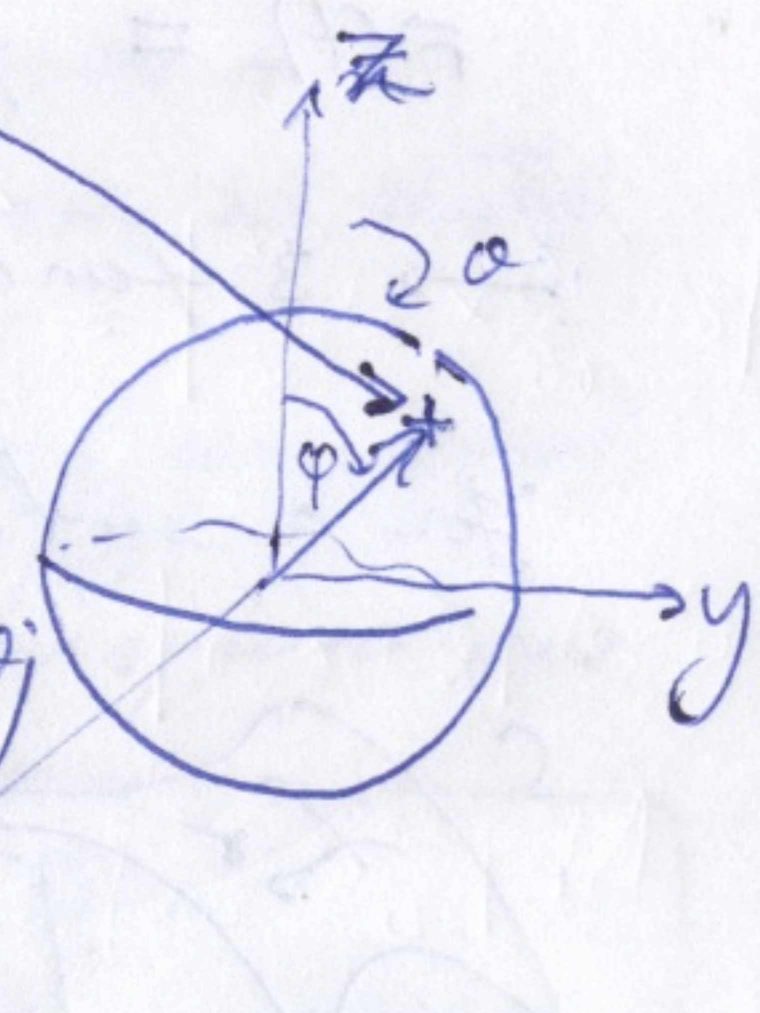




Example The sphere, 2-parameters

$(u, v) = (\theta, \varphi)$



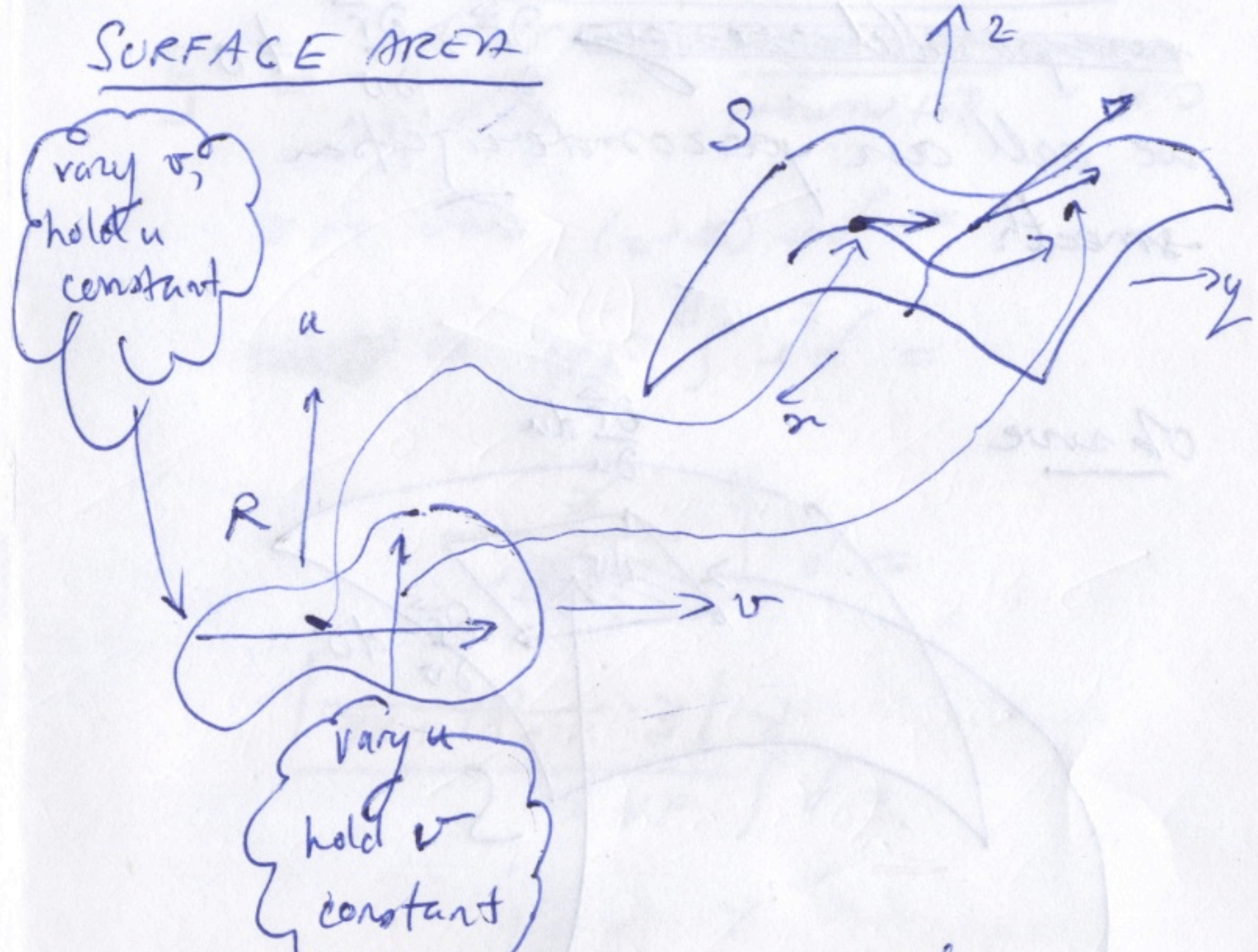


$\vec{r} = \cos\varphi\, k + \sin\varphi\cos\theta\, i + \sin\varphi\sin\theta\, j$

Here (φ, θ) have an obvious geometrical interpretations ~ angles of spherical coord.s

In fact, you can interpret (u, v) as coordinates for the surface S.

SURFACE AREA



straight lines in (u,v)-plane give curves in S.

$\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ give the tangent vectors parallel to these curves.

When $\frac{\partial \vec{r}}{\partial u} \equiv \vec{r}_u$ & $\frac{\partial \vec{r}}{\partial v} \equiv \vec{r}_v$

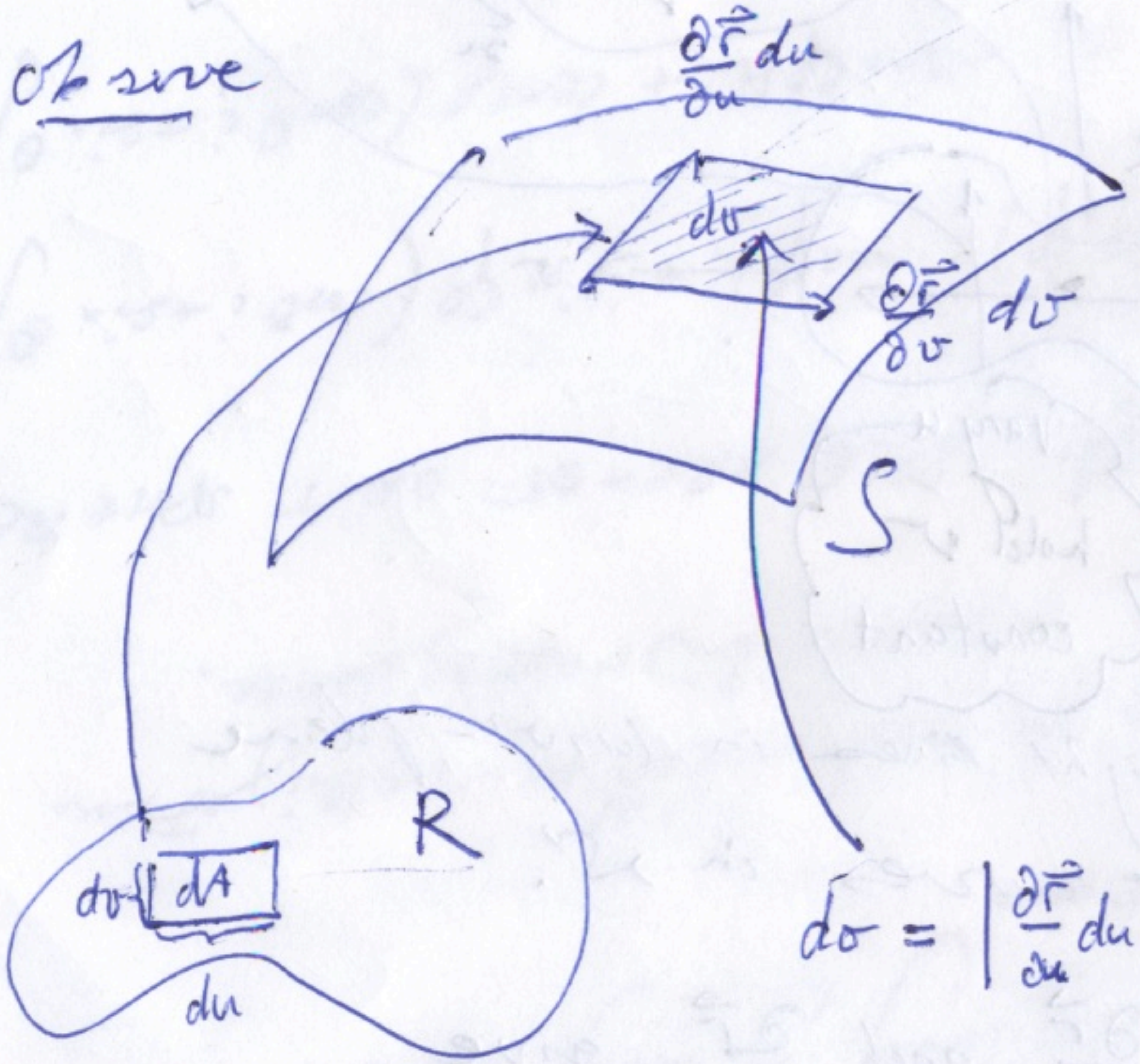
exist and are continuous &

~~are not parallel we say~~ $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq 0$

we call an parameterization

smooth.

Observe



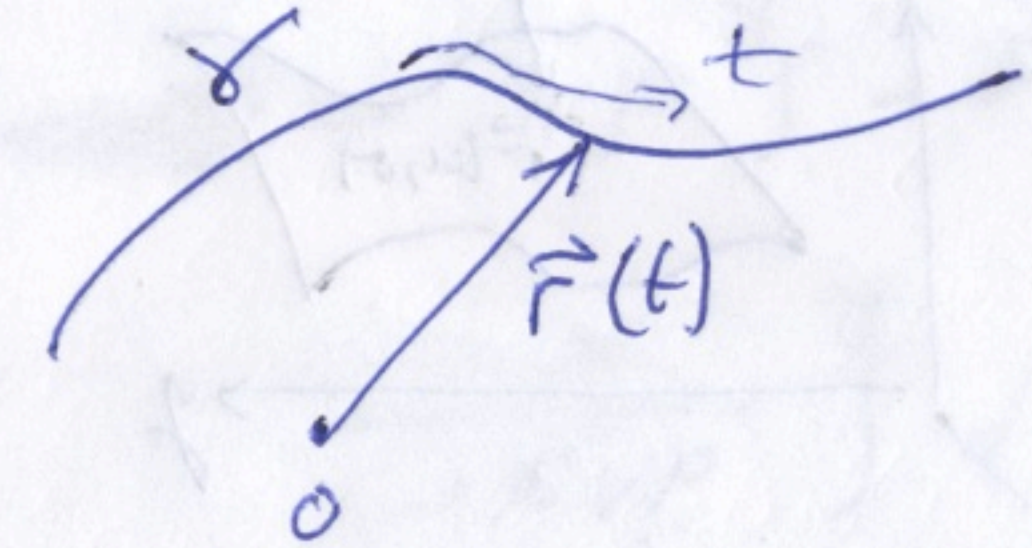
$$d\sigma = \left| \frac{\partial \vec{r}}{\partial u} du \times \frac{\partial \vec{r}}{\partial v} dv \right|$$

$$= \left| \vec{r}_u \times \vec{r}_v \right| du\, dv$$

$$= \left| \vec{r}_u \times \vec{r}_v \right| dA$$

PARAMETERIZED SURFACES (16.6)

Recall for curves, computations were often easy using a parameterization



even though many quantities depended only on the path γ, not any particular choice of parameterization $\vec{r}(t)$.

$$\vec{r}(t) = f(t)\,i + g(t)\,j + h(t)\,k$$

$\longrightarrow$ 3 functions of a single variable t

For a surface need 2 parameters

$$\boxed{\vec{r}(t) = f(u,v)\,i + g(u,v)\,j + h(u,v)\,k}$$

Surface area $2\pi r^2$

Mass $2\pi r^2 \rho$

By symmetry CoM is on z-axis

$\Rightarrow$ 1st moment in z-direction

$$\int_S z\rho\, d\sigma = \rho \int_R \frac{dA\, z}{|\cos\gamma|}$$

$$= \rho \int_R \frac{dA\, z}{|\vec{k}\cdot\vec{n}|}$$

$$= \rho \int_R \frac{dA\, z}{\left|\vec{k}\cdot \frac{xi+yj+zk}{r}\right|}$$

$$= \rho \int_R \frac{dA\, \cancel{z}\, r}{\cancel{z}}$$

$$= \rho r \int_R dA = $$

$$= \rho r (\pi r^2) \quad \text{area of } R$$

$$\text{CoM} = \frac{\text{1st moment}}{\text{total mass}} = \frac{\pi \rho r^3}{2\pi \rho r^2} = \frac{r}{2}$$

$\boxed{\tfrac{1}{2} \text{ way up } z\text{-axis is CoM}}$

Hence

$$\boxed{\text{AREA} = \int_S d\sigma = \int_R \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dA}$$

and similarly for integrals over functions.

Notice we need smooth parametrization to ensure this makes sense.

Example SFCE AREA OF UNIT SPHERE

$$\vec{r}(\theta,\varphi) = \cos\varphi\, k + \sin\varphi\,(\cos\theta\, i + \sin\theta\, j)$$

$$\frac{\partial \vec{r}}{\partial \varphi} = -\sin\varphi\, k + \cos\varphi\,(\cos\theta\, i + \sin\theta\, j)$$

$$\frac{\partial \vec{r}}{\partial \theta} = + \sin\varphi\,(-\sin\theta\, i + \cos\theta\, j)$$

$$\left|\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right| = \Big| \sin^2\varphi \sin\theta\, j + \frac{\cos\varphi\cos^2\theta}{\sin\varphi}\, k + \frac{\sin\varphi \sin^2\theta}{\cos\varphi}\, k + \sin^2\varphi\cos\theta\, i$$

$$= \sqrt{\sin^4\varphi\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \cos^2\varphi \sin^2\varphi}$$

$$= \sin\varphi$$