Recall curl $\vec{F} = \nabla \times \vec{F}$

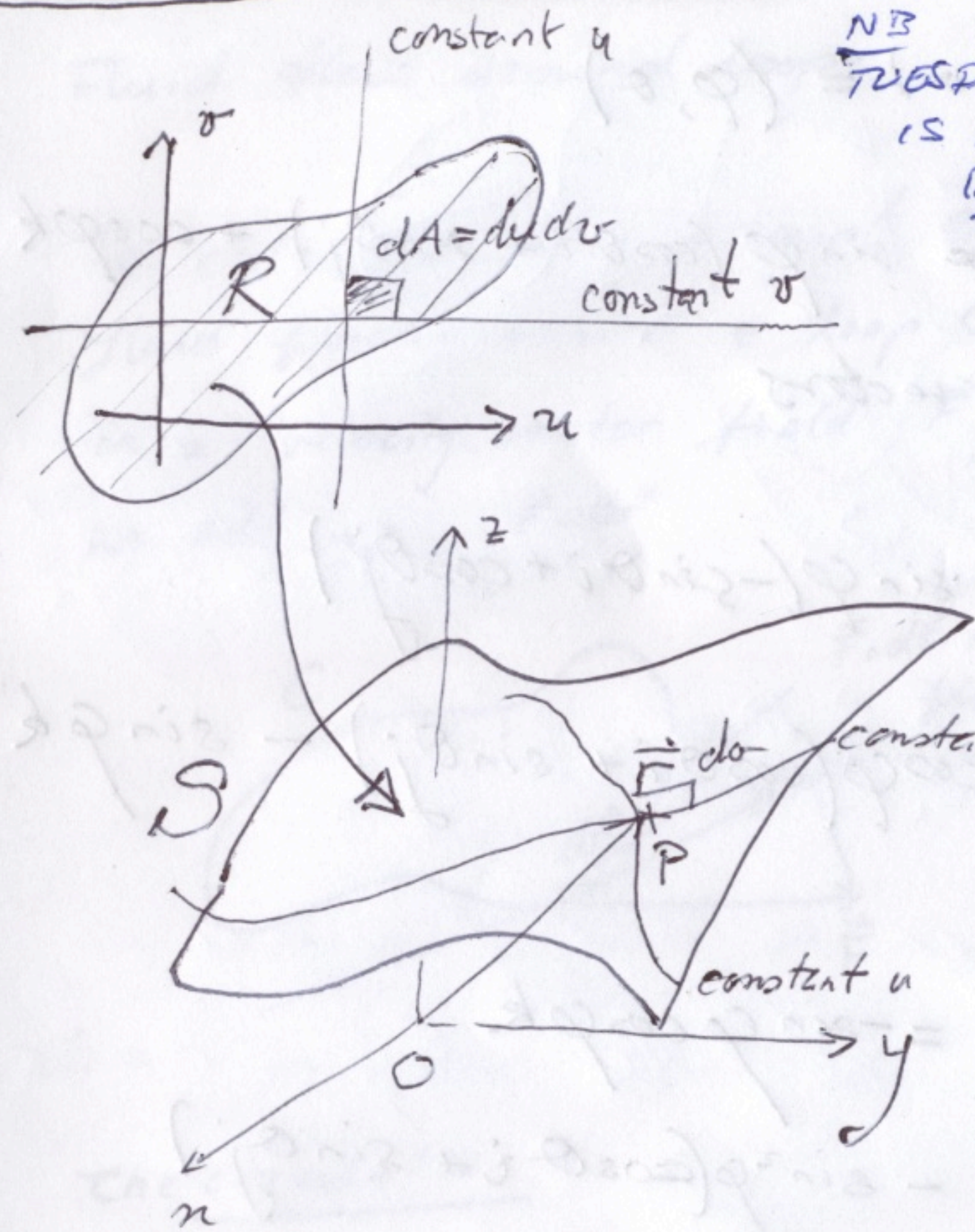
where

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Next time — an example.

Lecture #23 PARAMETERIZED SURFACES

NB NEXT TUESDAY IS REGRADE DEADLINE



$P \in S$, $\vec{r} = \overrightarrow{OP} = f(u,v)i + g(u,v)j + h(u,v)k$

Tangent vectors

$$\frac{\partial \vec{r}}{\partial u} \equiv \vec{r}_u, \qquad \frac{\partial \vec{r}}{\partial v} \equiv \vec{r}_v$$

Infinitesimal surface area

$$d\sigma = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du\, dv$$

<u>Example</u> The sphere

$$(u,\sigma) = (\varphi,\theta)$$

$$\vec{r}(\varphi,\theta) = \sin\varphi(\cos\theta\, i + \sin\theta\, j) + \cos\varphi\, k$$

Tangent vectors

$$\frac{\partial\vec{r}}{\partial\theta} = \sin\varphi(-\sin\theta\, i + \cos\theta\, j)$$

$$\frac{\partial\vec{r}}{\partial\varphi} = \cos\varphi(\cos\theta\, i + \sin\theta\, j) - \sin\varphi\, k$$

$$\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\varphi} = -\sin\varphi\cos\varphi\, k - \sin^2\varphi(\cos\theta\, i + \sin\theta\, j)$$

$$\left|\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\varphi}\right|^2 = \sin^2\varphi\cos^2\varphi + \sin^4\varphi = \sin^2\varphi$$

$$\Rightarrow d\sigma = \sin\varphi\, d\varphi\, d\theta$$

$$\text{SURFACE AREA} = \int_S d\sigma = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \sin\varphi = 2\pi \cdot 2 = 4\pi$$

Recall surface area of radius r sphere $= 4\pi r^2$

This answer could not possibly depend on our <u>names</u> for the coordinates & unit vectors i, j, k.

The important thing is that k is $\vec{n}$, the vector normal to the loop C. For small loops of area $d\sigma$, there is a well-defined normal $\vec{n}$

<u>Hence</u> for any loop $C = \partial S$ bounding a surface S

$$\underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\text{CIRCULATION}} = \int_S d\sigma\, \underbrace{(\mathrm{curl}(\vec{F})\cdot\vec{n})}_{\text{CIRCULATION DENSITY}}$$

Add up circulation for small loops 6

Circulation $= \oint_{C=\partial S} \vec{F}\cdot d\vec{r} = \int_S d\sigma$ (circulation density)

$\boxed{\text{LINE INTEGRAL}}$ $\boxed{\text{SURFACE INTEGRAL}}$

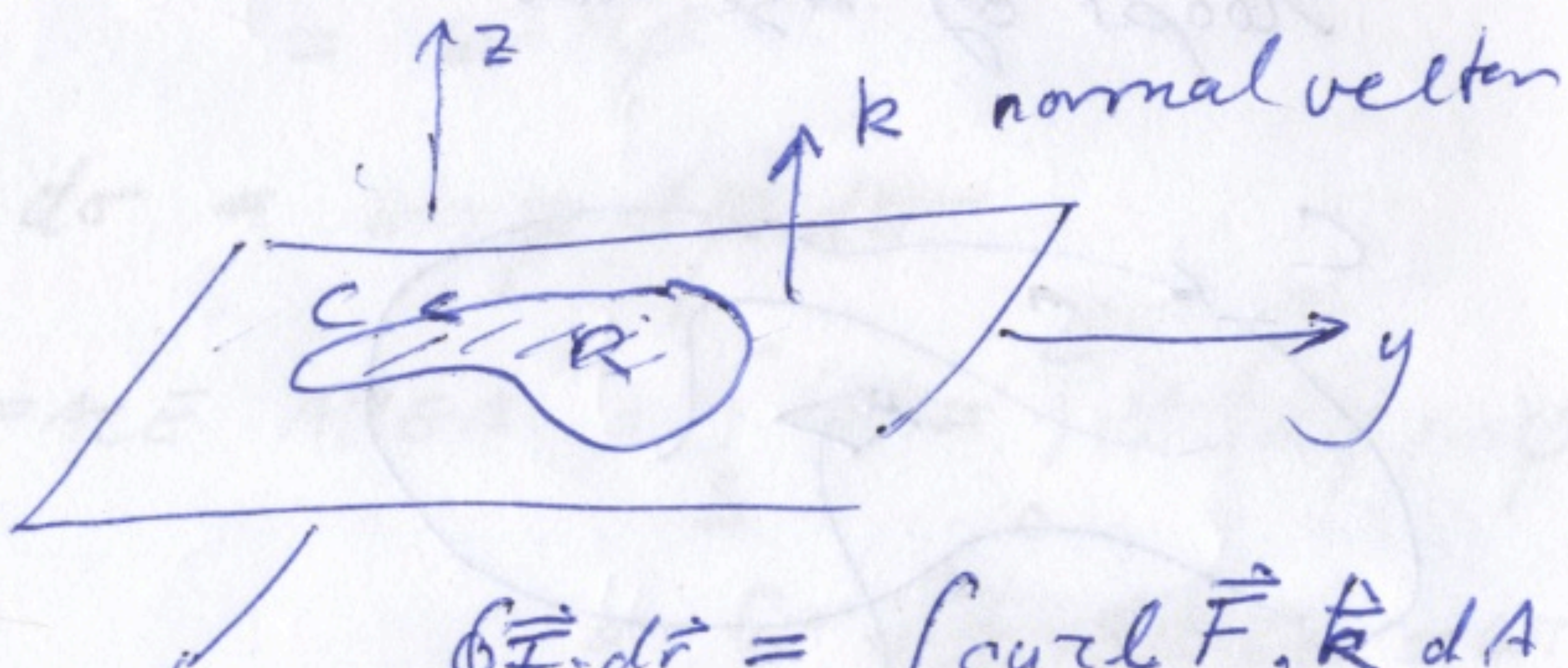
QUESTION

How do we find circulation density?

ANSWER

For loops in the plane we did it already:

GREENS' THEOREM (CIRCULATION FORM)

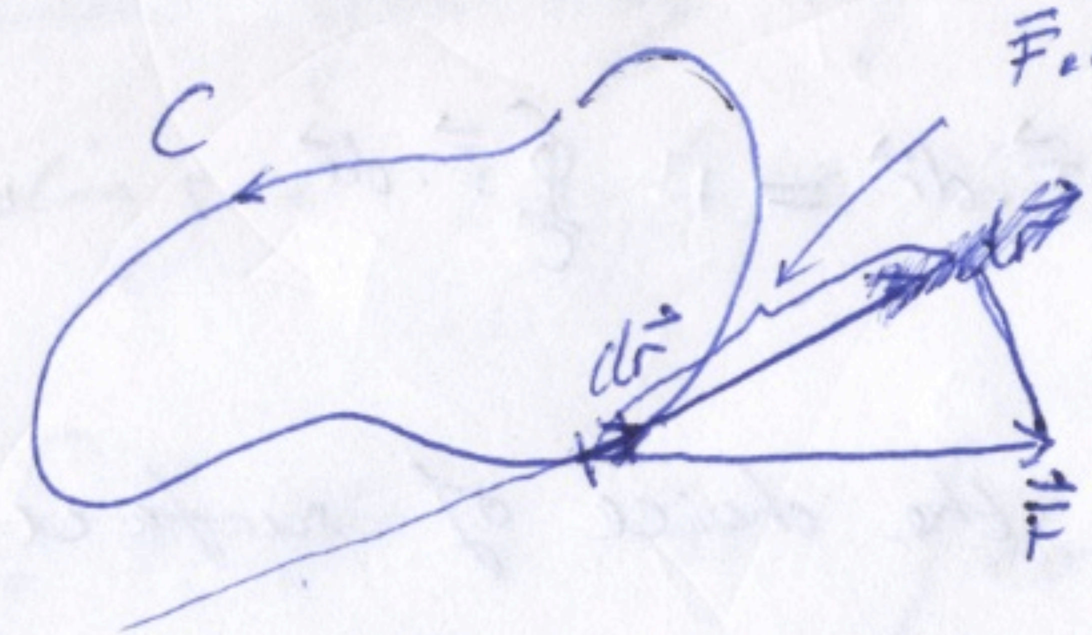


$$\oint_{C=\partial R} \vec{F}\cdot d\vec{r} = \int_R \underbrace{\text{curl}\,\vec{F}\cdot\vec{k}}_{\text{circulation density}}\, dA$$

STOKES' THEOREM 16.7 3

Fluid flows around loops

To calculate the amount of fluid flowing around a loop C in a velocity vector field $\vec{F}$ we add up $\vec{F}\cdot d\vec{r}$

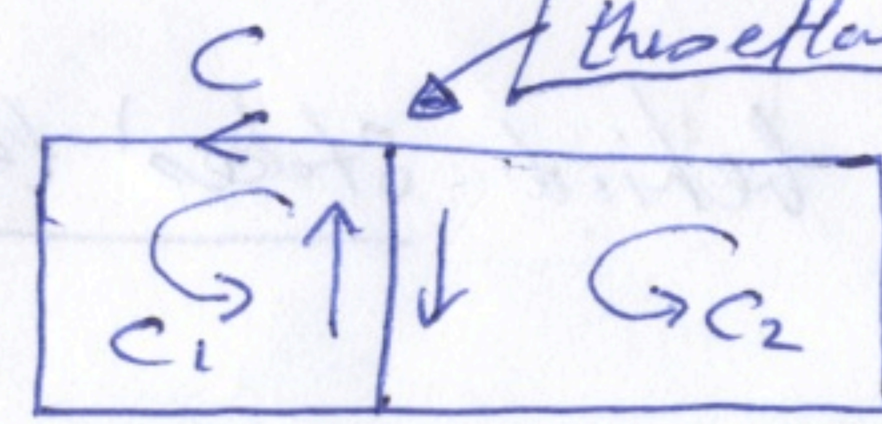


The circulation

$\oint_C \vec{F}\cdot d\vec{r}$ measures fluid flowing around C

Consider C built from two loops C_1 & C_2

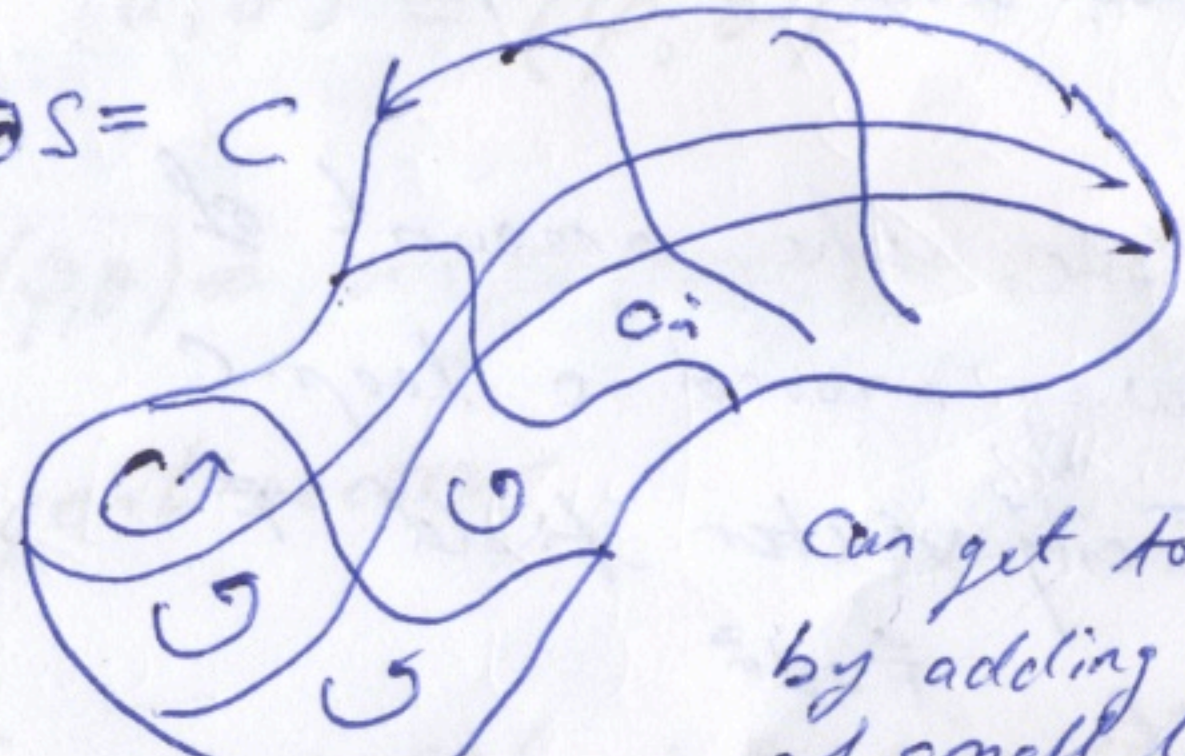
these flows cancel



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r}$$

Similarly for

$\partial S = C$

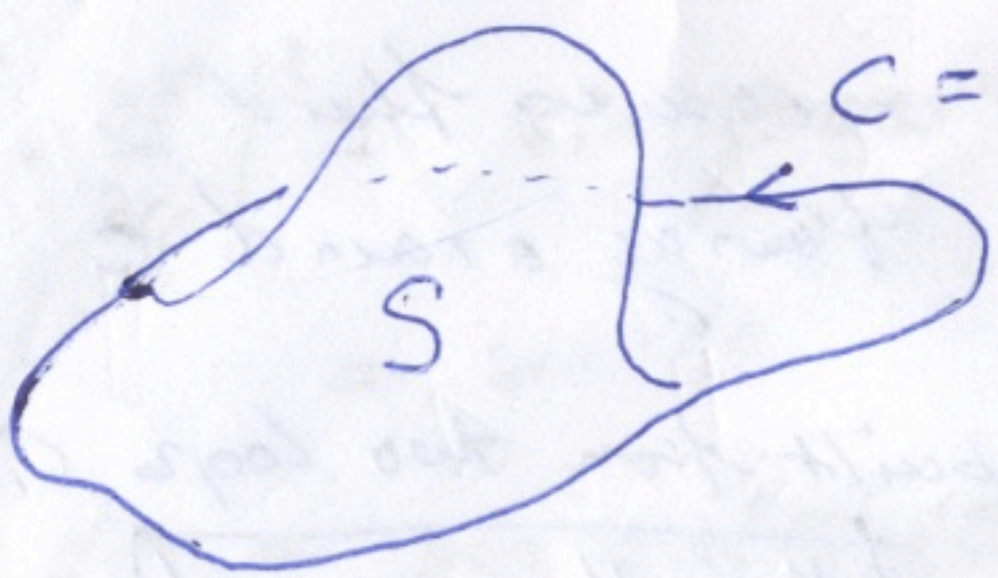


can get total circulation by adding circulations of <u>small loops</u>.

$$\sum_i \oint_{C_i} \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r}$$

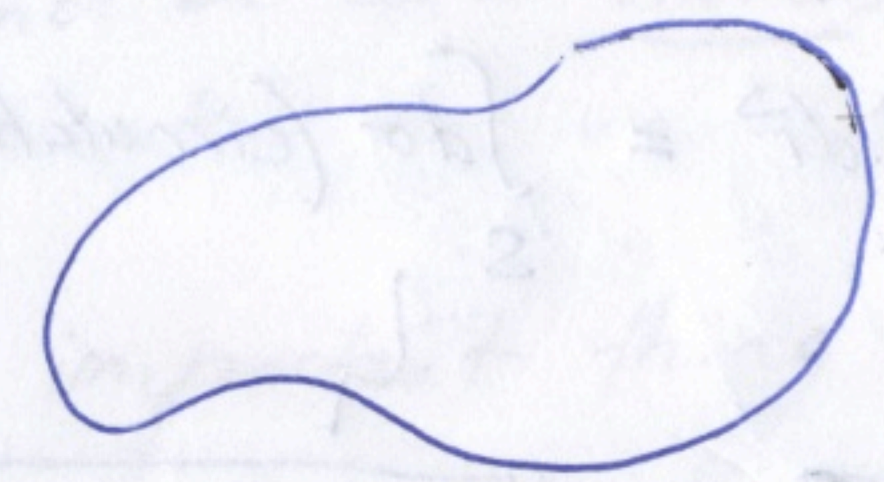
Moreover the choice of surface S shouldn't matter so long as $C = \partial S$

$C = \partial S$



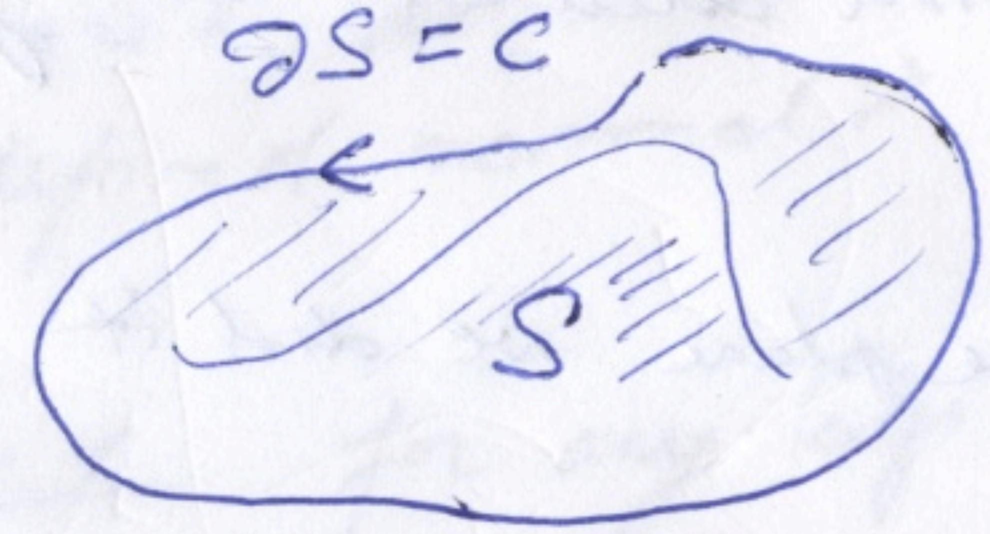
This is the idea behind <u>Stokes' theorem</u>.

(i) start with loop C



(ii) find some surface S where boundary $\partial S = C$

$\partial S = C$



(iii) Break S into infinitesimal loops of size $d\sigma$

C

$d\sigma$

(iv) Calculate circulation around infinitesimal loop

(circulation density) $\times d\sigma$

Depends on $\vec{F}$