Example

$$\vec{F} = yi - xj$$

Circulation around unit circle $x^2+y^2 =$ [redacted] $\quad z = 0$

LINE INTEGRAL

$$\vec{F}\cdot d\vec{r} = (yi - xj)\cdot(i\,dx + j\,dy)$$

$$= y\,dx - x\,dy$$

Choose parametrization

$$\vec{r} = \cos t\, i + \sin t\, j \qquad 0 \le t < 2\pi$$

$$\Rightarrow y\,dx - x\,dy = \sin t\, d(\cos t) - \cos t\, d(\sin t)$$

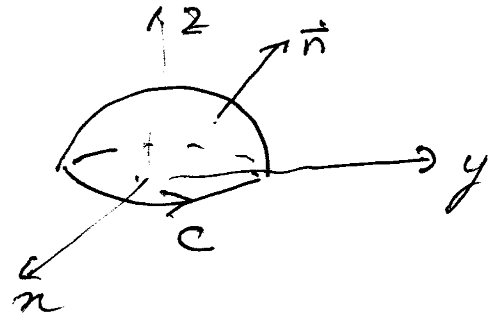
$$= -\sin^2 t\,dt - \cos^2 t\,dt$$

$$= -dt$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = -\int_0^{2\pi} dt = -2\pi$$

SURFACE INTEGRAL

$\boxed{\text{CHOOSE}}$ hemisphere $x^2+y^2+z^2 = 1$, $z \ge 0$ for S.



$$\vec{n} = \frac{xi + yj + zk}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{\nabla}\times\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(yi - xj)$$

$$= i\times j\,\frac{\partial}{\partial x}(-x) + j\times i\,\frac{\partial}{\partial y}(y)$$

$$= -2k$$

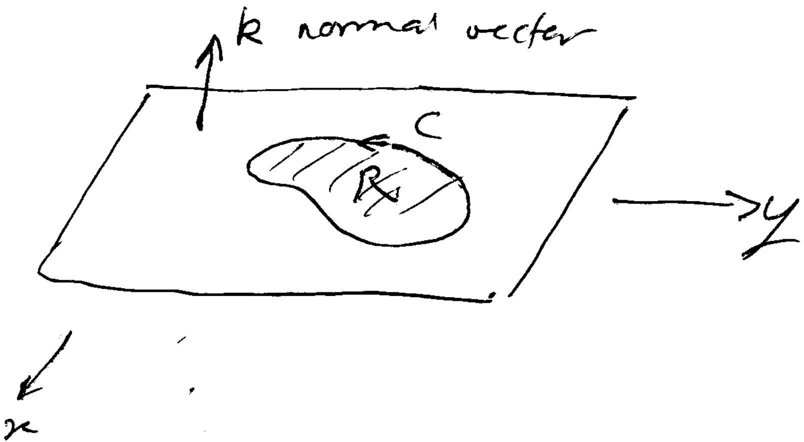
$$\nabla\times\vec{F}\cdot\vec{n} = \frac{-2z}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = \int_S \left(\frac{-2z}{\sqrt{x^2+y^2+z^2}}\right) d\sigma$$

QUESTION How to find circulation density?

ANSWER GREEN'S THEOREM (circulation form)

For loop in xy-plane



Circulation

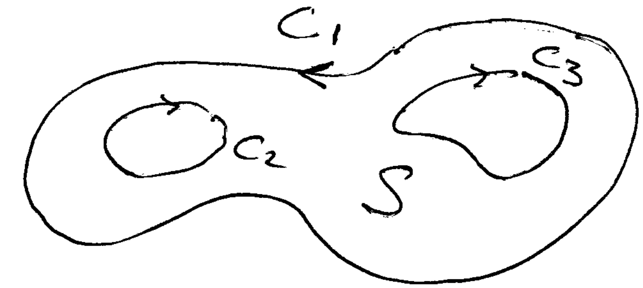
$$\oint_{C=\partial R} \vec{F}\cdot d\vec{r} = \int_R \operatorname{curl} \vec{F}\cdot\vec{k}\, dA$$

Hence $\operatorname{curl} \vec{F}\cdot\vec{k}$ is circulation density.

When loop is small, ie $d\sigma \to 0$, we can approximate it by a planar loop

Comments

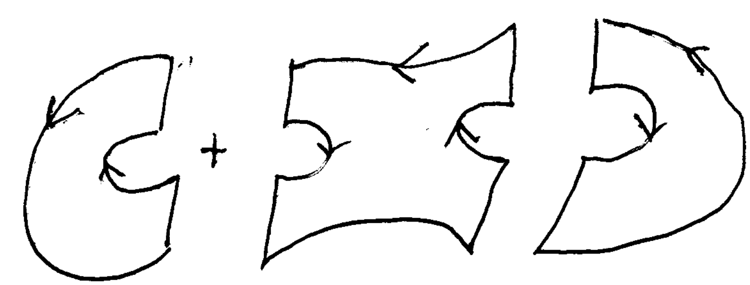
(1) If S has holes



$$\int_S d\sigma\,(\operatorname{curl} F)\cdot\hat{n} = \oint_{C_1} \vec{F}\cdot d\vec{r}$$

clockwise $\left\{ \begin{array}{l} + \oint_{C_2} \vec{F}\cdot d\vec{r} \\ + \oint_{C_3} \vec{F}\cdot d\vec{r} \end{array} \right.$

To see this consider



Notice If $\nabla \times \vec{F} = 0$

then $\vec{F}$ is conservative

~~Example~~

$\cancel{\vec{F} = M(x,y)\,i + N(x,y)\,j}$

$\cancel{\nabla \times \vec{F} =}$

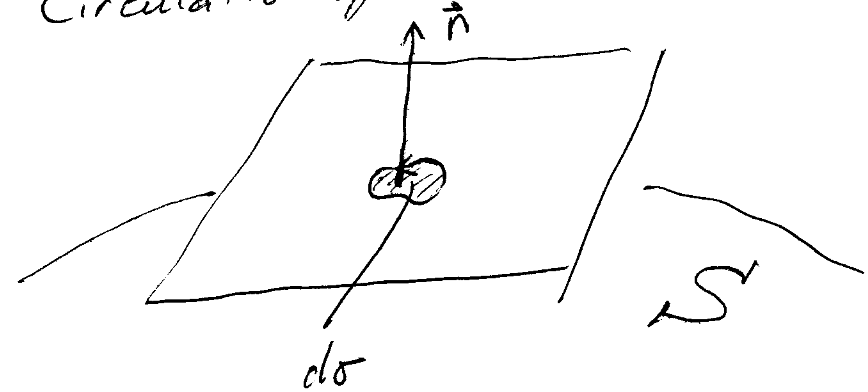
Notice that $\nabla \times \vec{F} = 0$ was exactly the mixed partial condition we applied earlier to test whether $\vec{F} = \nabla f$ for some potential f.

Lecture #24 STOKES' THEOREM

circulation:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S d\sigma \left(\begin{array}{c} \text{circulation} \\ \text{density} \end{array} \right)$$

Circulation of small loop $d\sigma$



Parameterize Hemisphere

$$z = \cos\varphi \qquad 0 \le \varphi < \frac{\pi}{2}$$

$$x = \sin\varphi\cos\theta \qquad 0 \le \theta < 2\pi$$

$$y = \sin\varphi\sin\theta$$

$$x^2+y^2+z^2=1 \qquad d\sigma = \sin\varphi\, d\varphi\, d\theta$$

$$\Rightarrow \int_S (\nabla\times F)\cdot\vec{n}\, d\sigma = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi\, d\varphi\,(-2\cos\varphi)$$

$$= 2\pi \int_{\cos\varphi=1}^{\cos\varphi=0} -d(\cos\varphi)(-2)\cos\varphi$$

$$= 2\pi \int_1^0 2\,du\,u \qquad u=\cos\varphi$$

$$= 4\pi \times \left(\frac{-1}{2}\right) = -2\pi$$

EXERCISE, TRY THIS PROBLEM
AGAIN FOR S THE UNIT
DISC WITH $C = \partial S$
[ITS EASIER!]

Hence, only need to
replace normal to xy plane k
by the unit normal $\vec{n}$ to surface S.

Hence circulation density

$$= (\text{curl}\,\vec{F})\cdot\vec{n}$$

Stokes theorem

$$\boxed{\oint_{C=\partial S} \vec{F}\cdot d\vec{r} = \int_S (\text{curl}\,\vec{F})\cdot\vec{n}\, d\sigma}$$

(counter clockwise)

Note S needs to be oriented!