Lecture #26

Review Session

Tuesday June 9

1 – 3pm

Medical Sciences Bldg 1C

Room 180

Practice Exam → in lieu of office hours

See homepage

Exam ~ 6 questions

~ focus on Homework 9
16.5 — 16.8

~ otherwise comprehensive

~ work through examples done in class!

Total mass of fluid inside $S = \partial D$

$$M = \int_D dV \rho$$

Rate fluid leaves S is

(i) $\dfrac{dM}{dt}$

(ii) $\displaystyle\int_S d\sigma\, \vec{F} \cdot \vec{n}$

These must be equal

divergence theorem

$$\frac{dM}{dt} = \int_S d\sigma\, \vec{F} \cdot \vec{n} = \int_D dV \operatorname{div} \vec{F}$$

Yet

$$\frac{dM}{dt} = \int_D dV \frac{\partial \rho}{\partial t}$$

since D is fixed (as is dV)

Hence

$$\int_D dV \frac{\partial \rho}{\partial t} = \int_D dV \operatorname{div} \vec{F}$$

Can we remove the integrations?

Yes — D was arbitrary so take it to be a very small cube of volume ε at any point in space.

Then our integral are approximately

$$\cancel{\varepsilon} \frac{\partial \rho}{\partial t} = \cancel{\varepsilon} \operatorname{div} F$$

i.e.

$$\boxed{\frac{\partial \rho}{\partial t} = \operatorname{div} \vec{F}}$$

Rate of change in density = Flux density

Incompressible fluid flows

$\longrightarrow$ constant density.

Hence $\boxed{\operatorname{div} \vec{F} = 0}$

Example

$$\vec{F} = \frac{GMm}{\rho^3}(xi + yj + zk)$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$\operatorname{div}\left(\frac{\vec{F}}{GMm}\right) = \operatorname{grad} \rho^{-3} \cdot (xi + yj + zk) + \rho^{-3} \cdot 3$$

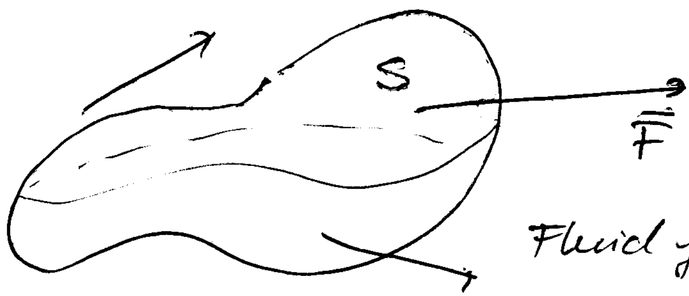
$$= \operatorname{grad} (\rho^2)^{-\frac{3}{2}} \cdot (xi + yj + zk) + \rho^{-3} \cdot 3$$

$$= -\frac{3}{2}(\rho^2)^{-\frac{5}{2}}(2xi + 2yj + 2zk) \cdot (xi + yj + zk) + \rho^{-3} \cdot 3 = 0$$

THE DIVERGENCE THEOREM

$$\boxed{\text{FLUX} = \oint_{S=\partial D} d\sigma\, \vec{F}\cdot\vec{n} = \int_D dV\, \operatorname{div}\vec{F}}$$

The continuity equation



Suppose $\vec{F}$ is the ~~rate~~ velocity of a fluid.

Then, for general $\vec{F}$, the amount of fluid in S must be changing.

Hence the fluid density

$$\rho = \rho(\vec{x}, t) \text{ per volume}$$

will depend on position $\vec{x}$ & time t.