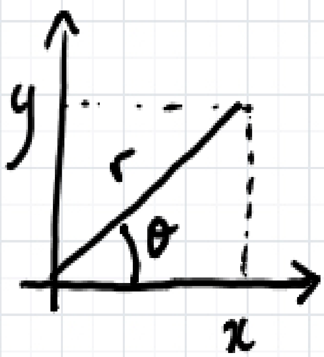
Lecture #5

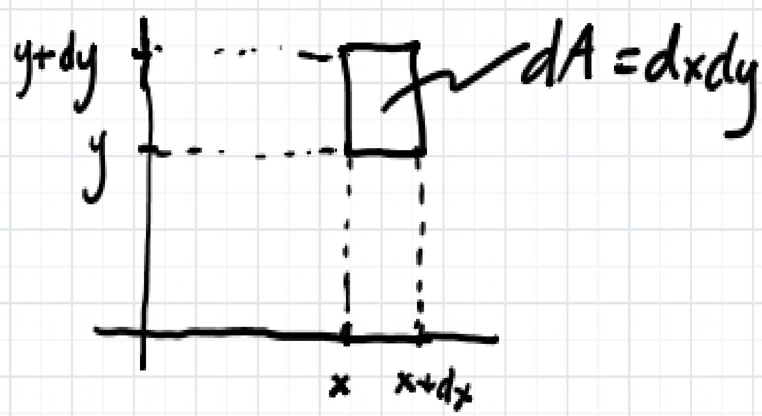
Relationship between Cartesian & Polar coordinates



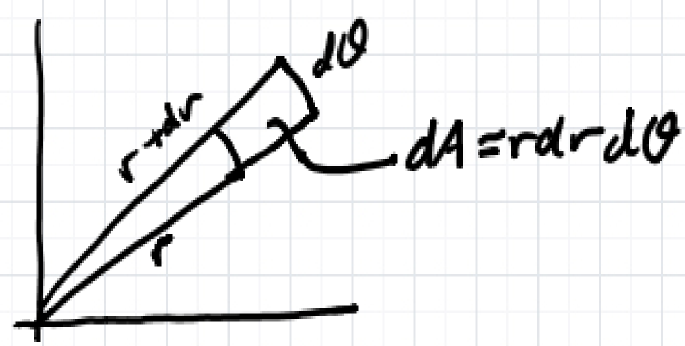
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Area element

$$dA = dx\,dy$$



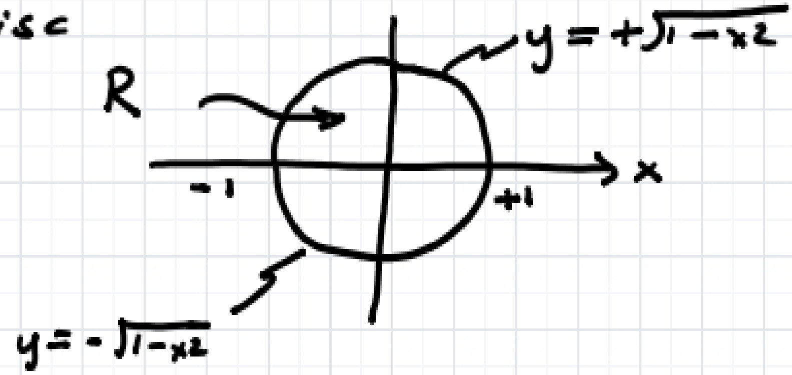
$$dA = r\,dr\,d\theta$$



Example Compute

$$I = \int_{-1}^{1} dx\, e^{-x^2} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, e^{-y^2}$$

Observe this is an area integral over the unit disc



and $dA = dx\,dy$, $f(x,y) = e^{-x^2-y^2}$

It could be easier in polar coordinates

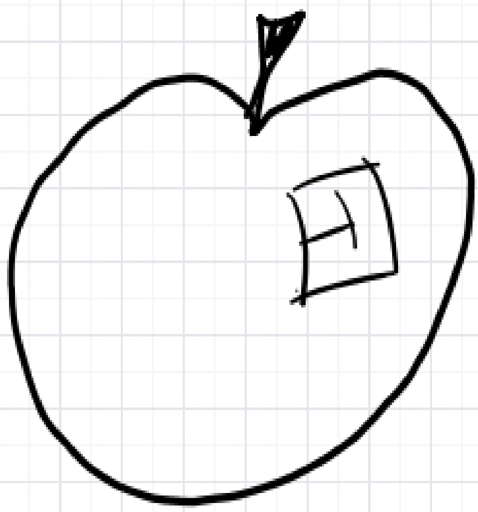
because $e^{-x^2-y^2} = e^{-(r\cos\theta)^2 - (r\sin\theta)^2} = e^{-r^2}$

$$\Rightarrow I = \int_0^{2\pi} d\theta \int_0^1 r\,dr\, e^{-r^2} = 2\pi \int_0^1 \tfrac{1}{2} d(r^2)\, e^{-r^2}$$

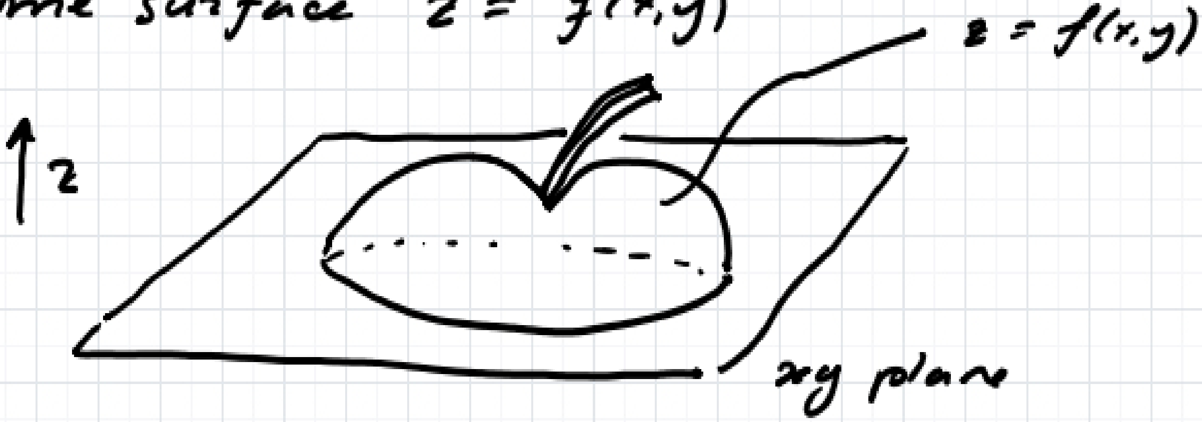
$$= 2\pi \cdot \tfrac{1}{2} \left[-e^{-r^2}\right]_{r^2=0}^{r^2=1} = \pi \left[\tfrac{1}{e} - 1\right]$$

VOLUME INTEGRALS (Ch. 15.4)

Suppose you wanted to compute the VOLUME of some solid



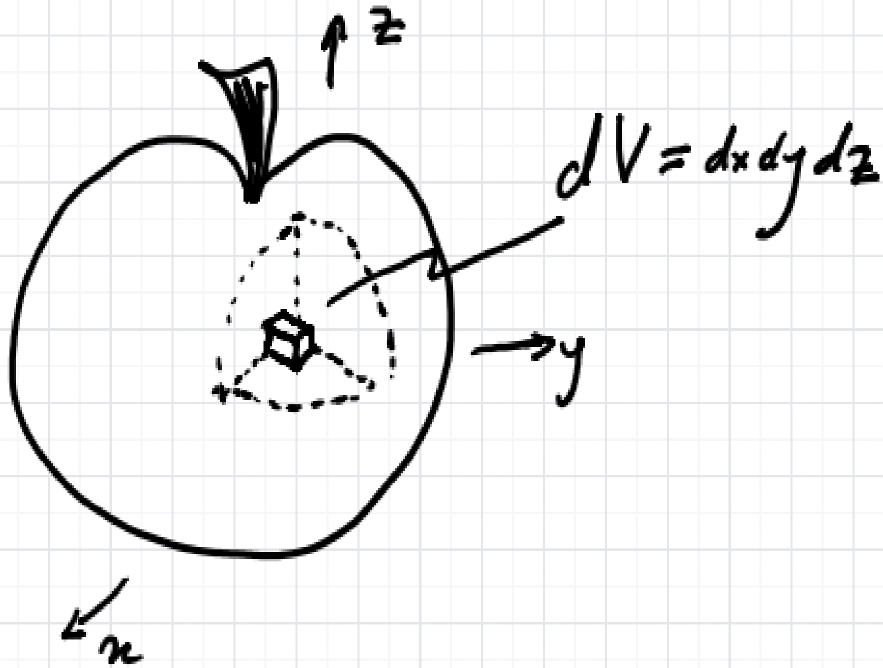
You could write it as volume under some surface $z = f(x,y)$



and similarly for the lower portion of the apple — this might be <u>tedious</u>!

Alternatively divide apple into infinitesimal pieces and add them up

VOLUME = $\int_D$



"closed bounded domain"

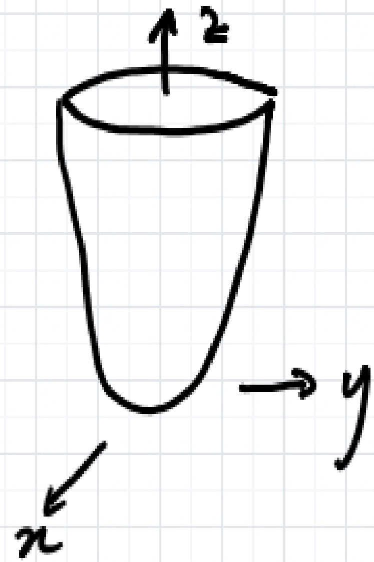
<u>i.e.</u> Integrate infinitesimal volume element against the function $f(x,y,z)=1$

$$\boxed{\text{VOLUME} = \int_D dV = \int_D dx\,dy\,dz}$$

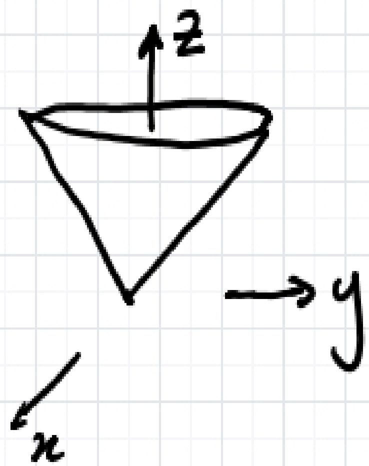
This will require <u>three</u> integrations so is often called a <u>triple integral</u>.

Example Find the volume between the

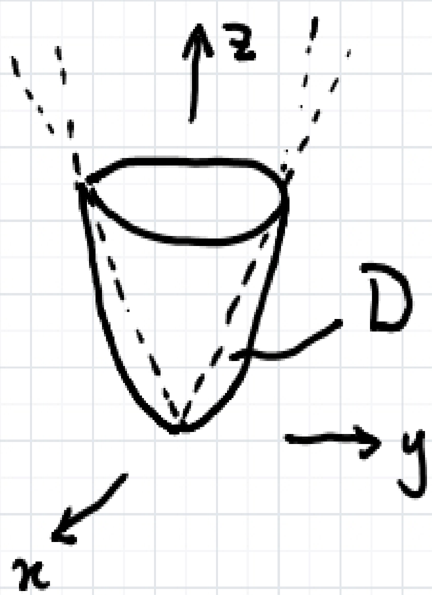
paraboloid $z = x^2 + y^2$



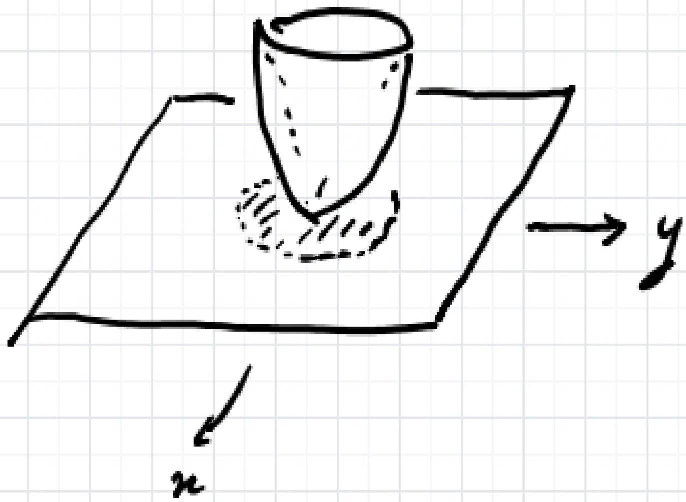
and cone $z^2 = x^2 + y^2$



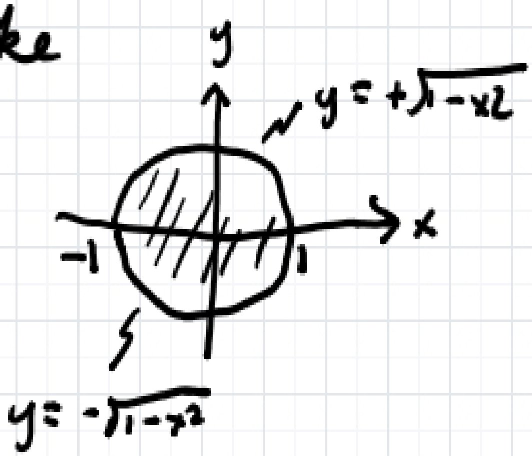
① Sketching and describe region



"Shadow" in xy plane (say)



Looks like



because cone & parabola intersect at circle radius 1 in $z=1$ plane.

$\Rightarrow \quad -1 \leq x \leq 1$

$-\sqrt{1-x^2} \leq y \leq \sqrt{1+x^2}$

but must find z-limits of integration!

Above (x,y), $\quad \underbrace{x^2+y^2}_{\text{paraboloid}} \le z \le \underbrace{\sqrt{x^2+y^2}}_{\text{cone}}$

② Choose order of integration (c.f. slices)

Here z, y, x

③ Set up integral

$$\text{VOLUME} = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} dy \int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz$$

$$= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \left[\sqrt{x^2+y^2} - (x^2+y^2)\right]$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,dr \left[r - r^2\right] = 2\pi\left[1-\tfrac{1}{2}\right] = \pi$$

$\boxed{\text{Polar coord.s easier}}$

Integrating functions over volumes

Just as for regions in the plane if $f : D \to \mathbb{R}$ is some function of a domain D we may consider

$$\int_D dV\, f$$

These integrals have applications analogous to area integrals. For example the average value of f over the domain D is

$$\boxed{\bar{f} \equiv \langle f \rangle \equiv \frac{\int_D f\, dV}{\int_D dV} = \frac{\int_D f\, dV}{\text{volume of } D}}$$