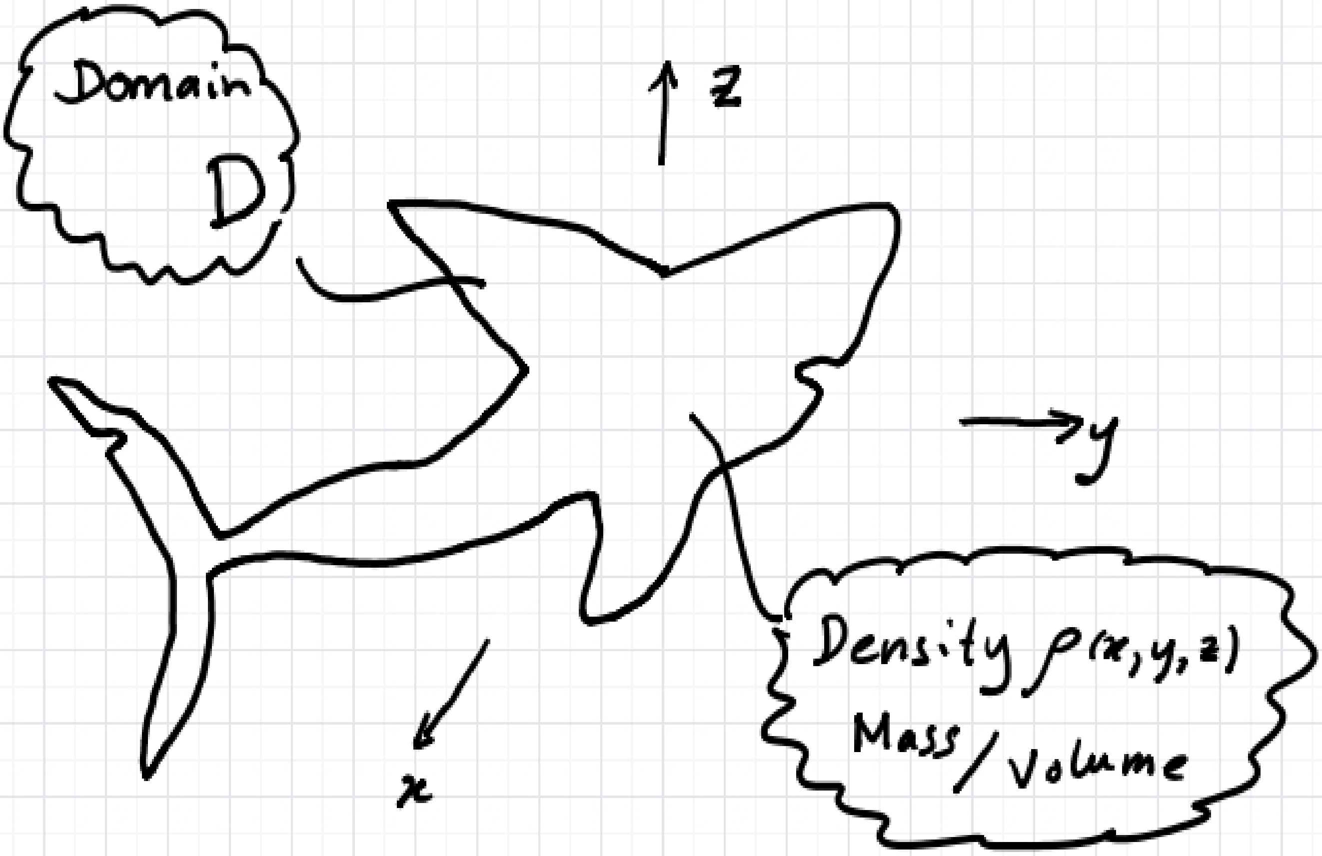
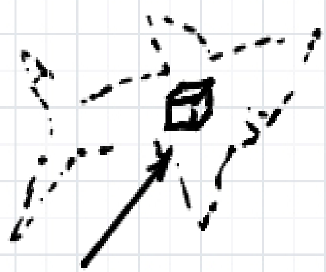
LECTURE #6

APPLICATIONS OF VOLUME INTEGRALS:

MASS & MOMENTS (ch. 15.5)

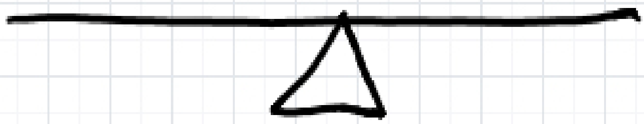


Total mass = $\int_D$  $\cdot\, \rho(x,y,z)$

$dV = dx\,dy\,dz$

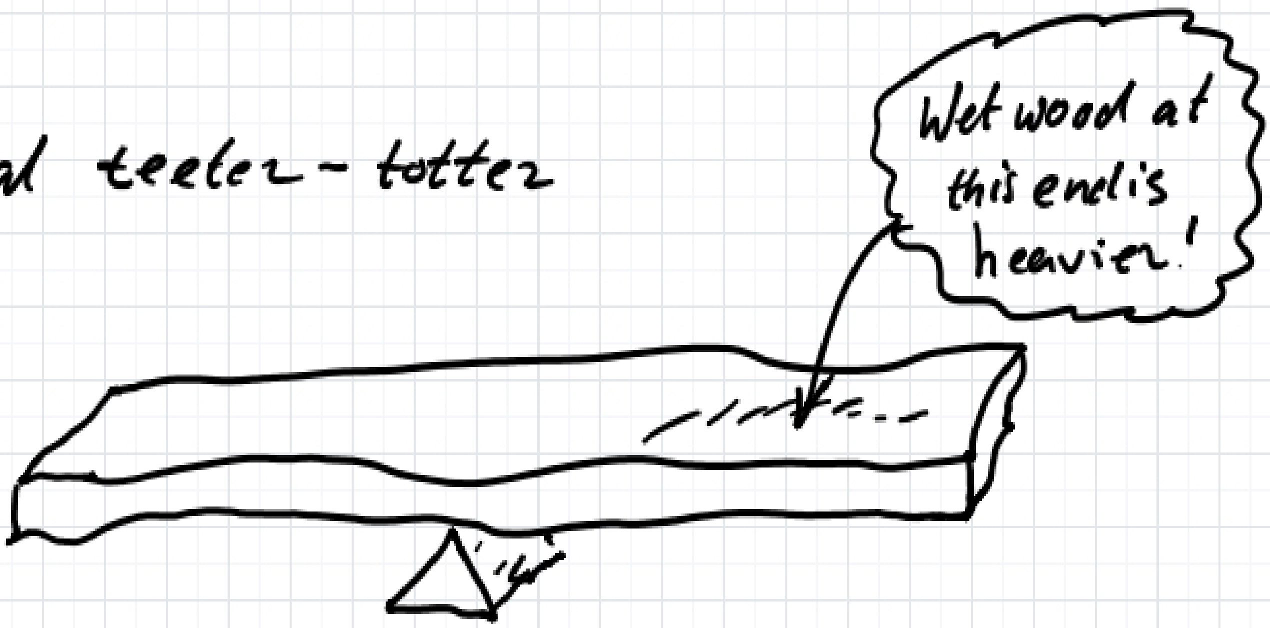
$= \int_D \rho\, dV$ b/c mass of infinitesimal volume dV is $\rho\, dV$.

FIRST MOMENTS



is too idealistic.

Real teeter-totter



where will it balance?

Centre of mass

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int_D x \rho \, dV}{\int_D \rho \, dV}, \frac{\int_D y \rho \, dV}{\int_D \rho \, dV}, \frac{\int_D z \rho \, dV}{\int_D \rho \, dV} \right)$$

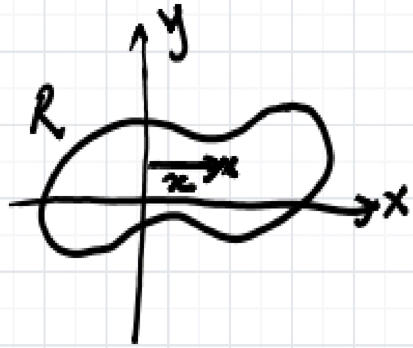
Note: If we could integrate over vectors such as

$$\vec{r} = (x, y, z)$$

then the centre of mass

$$\boxed{\vec{\bar{r}} = \frac{1}{\text{mass}} \times \int_D \vec{r} \rho \, dV}$$ [Later Ch. 3]

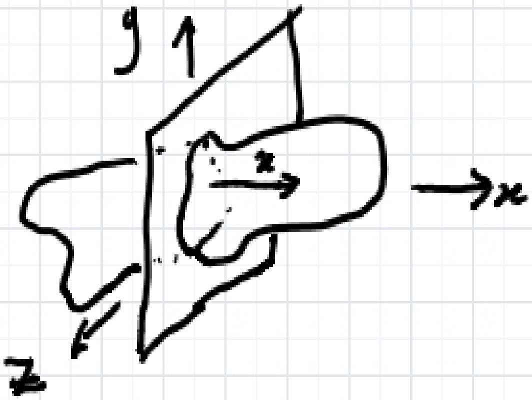
Note: Recall



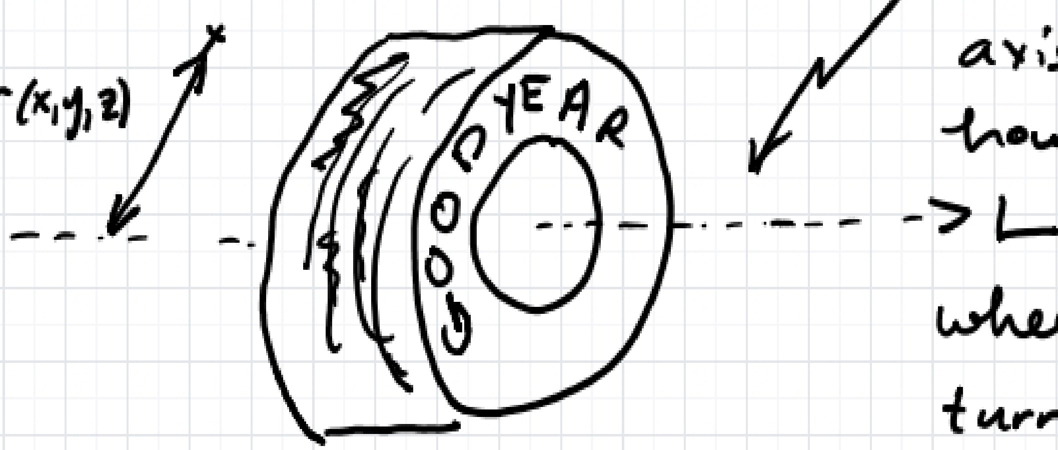
$\int_R x \rho \, dA$

was the <u>1^{st} moment about y-axis</u>.

We now have $\int_D x \rho \, dV$ is <u>1^{st} moment about yz-plane</u>



Moments of inertial are still defined about axis of rotation

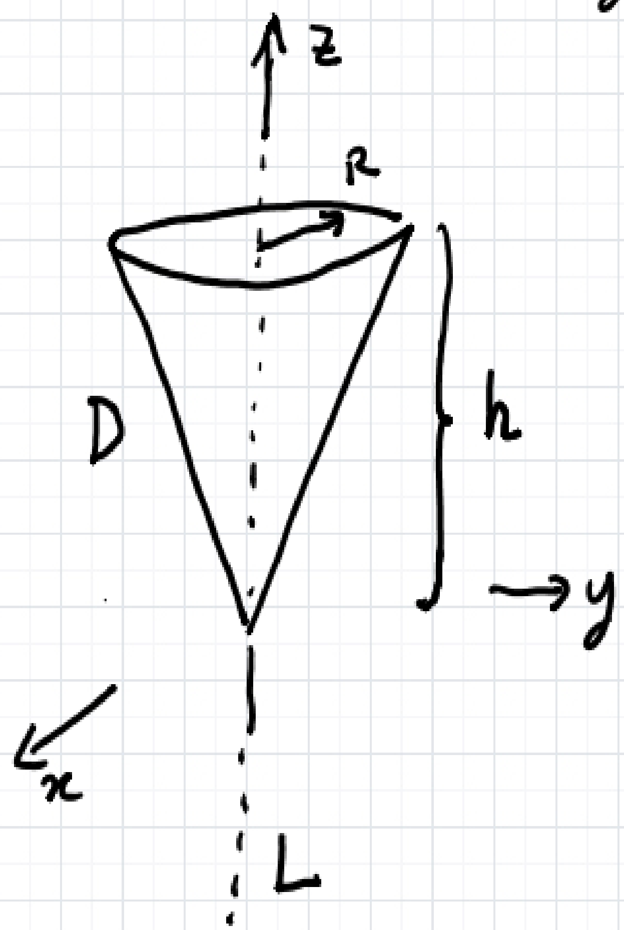


Distance $r(x,y,z)$ of mass from axis determines how easier wheel is to turn

Moment of inertia about L

$$\boxed{I_L = \int r^2 \rho \, dV}$$

<u>Example</u> Find the moment of inertia of a mass M, uniform cone of height h radius R about its axis of symmetry.



Recall, volume of cone $= \frac{1}{3}(\text{Base Area}) \times \text{Height}$

$\Rightarrow$ Density $\rho = M/\left[\frac{1}{3}\pi R^2 h\right]$

Describe domain:

$$\begin{cases} -R \le x \le R \\ -\sqrt{R^2-x^2} \le y \le \sqrt{R^2-x^2} \\ \frac{h}{R}\sqrt{x^2+y^2} \le z \le h \end{cases}$$

Describe function

$$r^2\rho = (x^2+y^2)\cdot \frac{M}{\frac{1}{3}\pi R^2 h}$$

Set up integral

$$I_L = \frac{M}{\frac{1}{3}\pi R^2 h}\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy\, (x^2+y^2)\int_{\frac{h}{R}\sqrt{x^2+y^2}}^{h} dz$$

$$= \frac{M}{\frac{1}{3}\pi R^2 h}\int_{-R}^{R} dx \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-x^2}} dy \left[h(x^2+y^2) - \frac{h}{R}(x^2+y^2)^{3/2}\right]$$

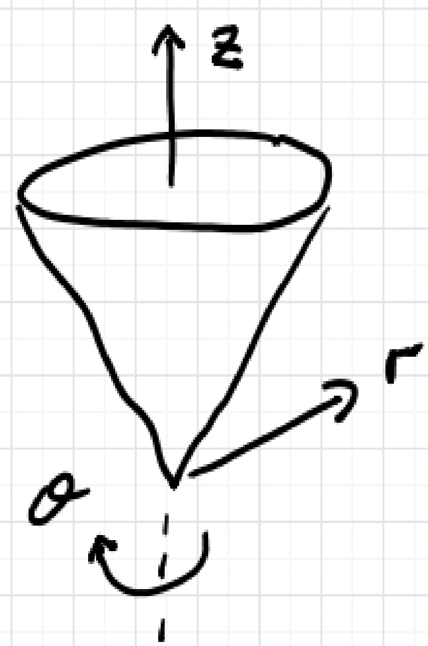
Trick: Use polar coordinates for xy integral

$$\Rightarrow I_L = \frac{M}{\frac{1}{3}\pi R^3} \int_0^{2\pi} d\theta \int_0^R r\,dr \left[hr^2 - \frac{h}{R} r^3 \right]$$

$$= \frac{M}{\frac{1}{3}\pi R^3} \, 2\pi \left[h \frac{1}{4} R^4 - \frac{h}{R} \frac{1}{5} R^5 \right]$$

$$= \frac{3}{10} MRh$$

<u>OBSERVE</u> It would clearly be nicer to do this problem in <u>cylindrical coordinates</u>



Integrating functions over volumes

Just as for regions in the plane if $f : D \to \mathbb{R}$ is some function of a domain D we may consider

$$\int_D dV\, f$$

These integrals have applications analogous to area integrals. For example the average value of f over the domain D is

$$\boxed{\bar{f} \equiv \langle f \rangle \equiv \frac{\int_D f\, dV}{\int_D dV} = \frac{\int_D f\, dV}{\text{volume of } D}}$$