LECTURE #9

VARIABLE CHANGES — THE JACOBIAN (ch 15.7)

Recall that in rectangular coordinates (x,y)

$$dA = dx\,dy$$

but changing to polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

we found (using some trigonometry)

$$dA = r\,dr\,d\theta$$

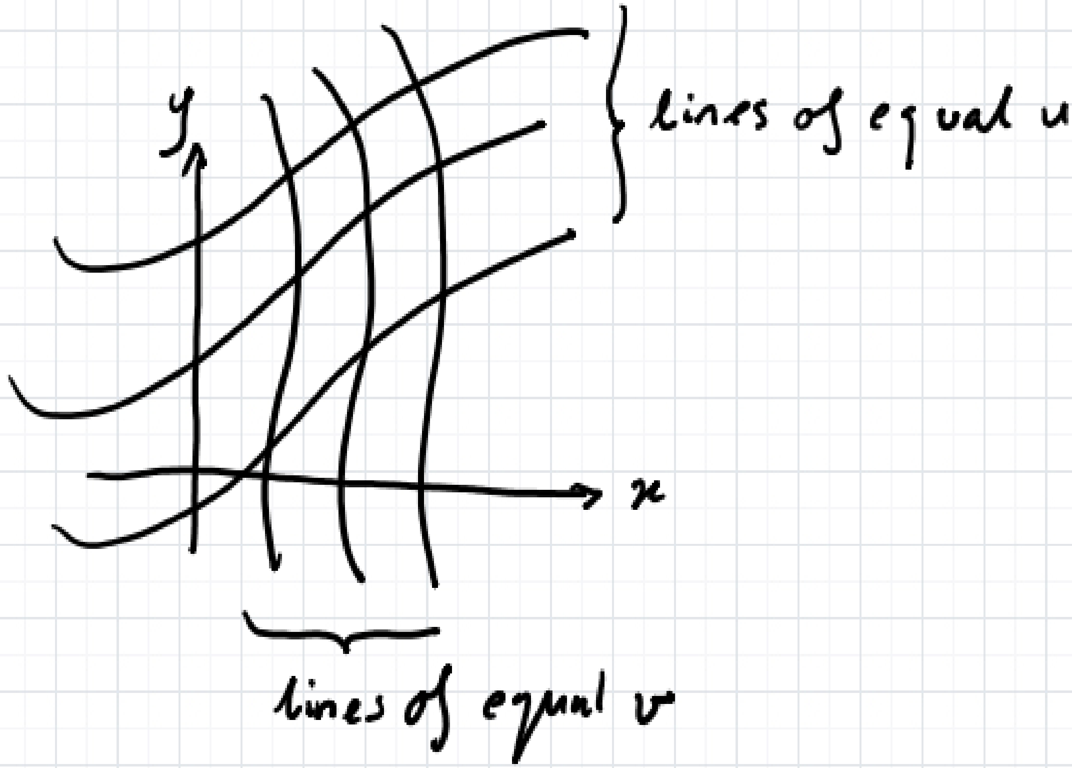
Notice $\left|\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right| dr\,d\theta = \left|\cos\theta \cdot r\cos\theta - (-r\sin\theta)\cdot\sin\theta\right| dr\,d\theta$

$= r(\cos^2\theta + \sin^2\theta)\,dr\,d\theta = r\,dr\,d\theta$

$= dA$ THIS IS NOT A FLUKE!

<u>Consider</u> New coordinates (u, v)

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

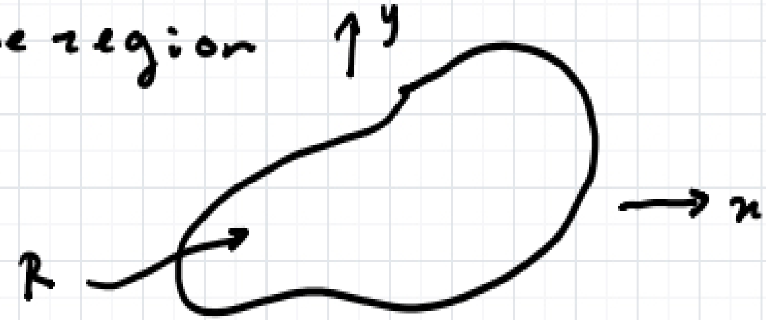


Form the matrix and take determinant

$$J(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Then $dA = du\,dv\,|J(u,v)|$

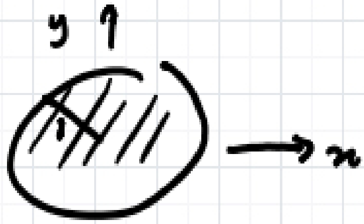
The region



in the plane is unchanged when different coordinates are used.

BUT does look different in the Cartesian uv-plane!

Example



is described by $0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

which looks like

Another example

The Earth

square sheets of paper used to describe a sphere!

Example

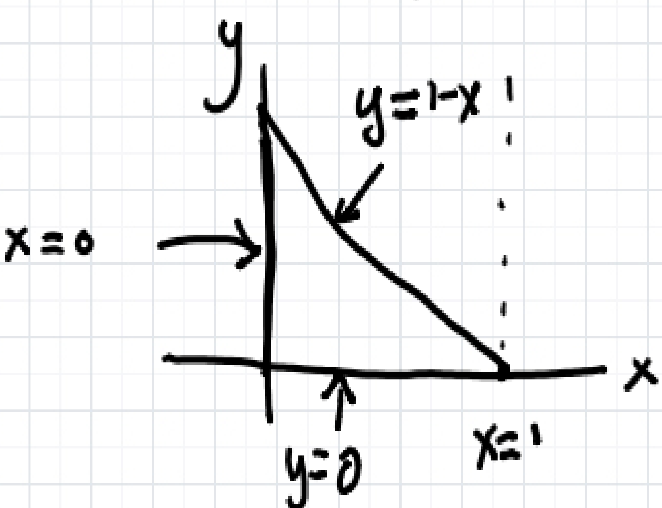
$$I = \int_0^1 dx \int_0^{1-x} dy \sqrt{x+y}\,(y-2x) = \int_R dA\, f$$

New variables $\begin{cases} u = x+y \\ v = y-2x \end{cases} \Rightarrow \begin{array}{l} x = \dfrac{u-v}{3} \\ y = \dfrac{2u+v}{3} \end{array}$

$$J(u,v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \left| \frac{1}{3}\cdot\frac{1}{3} - \frac{-1}{3}\cdot\frac{2}{3} \right| = \frac{1}{3} \Rightarrow dA = \frac{1}{3}\, du\, dv$$

Boundary of R

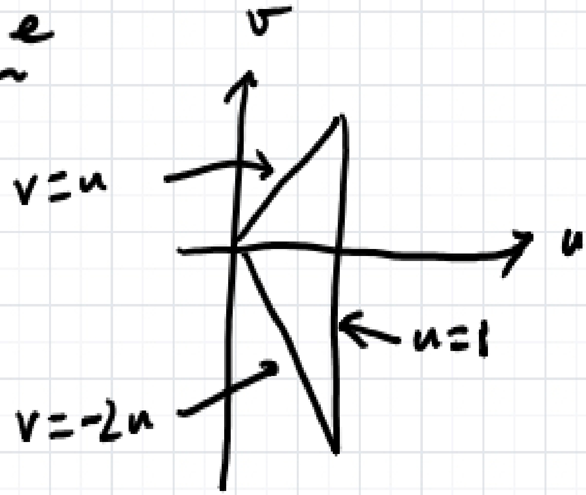


$x = 0 \Rightarrow \frac{u-v}{3} = 0$

$y = 0 \Rightarrow \frac{2u+v}{3} = 0$

$y = 1-x \Rightarrow u = 1$

uv-plane



Function $f = \sqrt{x+y}\,(y-2x)^2 = \sqrt{u}\, v^2$

Integral

$$I = \int_0^1 du \int_{-2u}^{u} dv \cdot \underset{J}{\frac{1}{3}} \cdot \sqrt{u}\, v^2$$

$$= \int_0^1 du \; \frac{1}{3} \left[\frac{1}{3} v^3 \right]_{-2u}^{u} u^{\frac{1}{2}}$$

$$= \frac{1}{9} \int_0^1 du \; \left[u^3 + 8u^3 \right] u^{\frac{1}{2}} = \int_0^1 u^{\frac{7}{2}} = \frac{9}{2}$$

CHANGING COORDINATES IN VOLUME INTEGRALS

Again Jacobian is given by determinant.

Example
$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\cos\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix}$$

$$= \rho^2 \det \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} \underset{\text{hard work}}{=} \rho^2 \sin\varphi$$

factor out ρ

As before.

Integrating functions over volumes

Just as for regions in the plane if $f : D \to \mathbb{R}$ is some function of a domain D we may consider

$$\int_D dV\, f$$

These integrals have applications analogous to area integrals. For example the average value of f over the domain D is

$$\boxed{\bar{f} \equiv \langle f \rangle \equiv \frac{\int_D f\, dV}{\int_D dV} = \frac{\int_D f\, dV}{\text{volume of } D}}$$