250A Homework 1

Due Monday October 11

Question 1 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator. Show that T has 1 and 2 dimensional invariant subspaces.

Question 2 Let M be the space of 3 matrices. What is $\dim(M)$? Now define the linear operator $T : M \to M$ by

$$M \ni m \stackrel{T}{\longmapsto} \frac{1}{2}\left[\begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} m + m \begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix}\right] .$$

Compute $\det T$.

Question 3 *Van der Monde Determinant.* Let A be the $(n \times n)$ matrix with entries

$$A_{ij} = (x_i)^j .$$

Show that $\det A = \prod_{i>j}(x_i - x_j)$.

Question 4 *(Anti)commutators.* Let V be a finite dimensional vector space. Show that the mapping

$$[\cdot,\cdot] : L(V) \times L(V) \to L(V)$$

where

$$[\cdot,\cdot] : (M,N) \mapsto MN - NM \equiv [M,N] ,$$

obeys the Leibnitz rule $[M,NR] = [M,N]R + N[M,R]$. In addition, verify the Jacobi identity

$$[M,[N,R]] + [N,[R,M]] + [R,[M,N]] = 0 .$$

Generalize the above laws to the mapping $\{\cdot,\cdot\} : (M,N) \mapsto MN + NM \equiv \{M,N\}$. Include also new rules which mix both operations.

Question 5 *Baker Campbell Hausdorff Formula.* Let V be a finite dimensional vector space and $M, N \in L(V)$. Show that

$$\exp(M+N) = \exp(M)\exp(N)\exp(-\frac{1}{2}[M,N]) ,$$

if $0 = [M,[M,N]] = [N,[M,N]]$. *Hint: Develop and solve a differential equation for* $R(\lambda) \equiv \exp(\lambda M)\exp(\lambda N) \in L(V)$.