Homework 5

due November 6, 2001

(1) A subgroup H of a group G is called characteristic in G, denoted H char G, if $\sigma(H) = H$ for all $\sigma \in \mathrm{Aut}(G)$. Show the following:
  (a) Characteristic subgroups are normal.
  (b) If H char K and $K \trianglelefteq G$, then $H \trianglelefteq G$.

(2) Let G be a group.
  (a) Show that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. The quotient group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the group of outer automorphisms of G.
  (b) If G is cyclic of order n, compute the orders of $\mathrm{Inn}(G)$ and $\mathrm{Out}(G)$.

(3) Let G be a group of order 20 and let H be a subgroup of order 5. Prove that G contains a proper normal subgroup N that contains H.

(4) If G is a finite p-group, $H \trianglelefteq G$ and $H \neq \{1\}$, then $H \cap Z(G) \neq \{1\}$.

(5) If H is a normal subgroup of order p^k of a finite group G, then H is contained in every Sylow p-subgroup of G.

(6) How many elements of order 7 are there in a simple group of order 168?