250A Homework 10

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Question 1 (*Internal direct sum/product*) Show that if a ring $R = J_1 + \cdots + J_n$ and the ideals $J_i \cong R_i$ for rings R_i and $J_i \cap \sum_{j \neq i} J_j = \{0\}$, then $R \cong \otimes_{i=1}^{n} R_i$.

Solution Define $\varphi : \prod_{i=1}^{n} J_i = J_1 \times J_2 \times \cdots \times J_n \longrightarrow R$ by

$$\{a_i\} = (a_1, a_2, \ldots, a_n) \longmapsto \sum a_i = a_1 + a_2 + \cdots + a_n$$

and let $a_i, b_i \in J_i$ for $1 \leq i \leq n$.
(i) φ preserves additive structure: $\varphi(\{a_i\} + \{b_i\}) = \varphi(\{a_i + b_i\}) = \sum(a_i + b_i) = \sum a_i + \sum b_i = \varphi(\{a_i\}) + \varphi(\{b_i\})$.
(ii) φ preserves multiplicative structure: $\varphi(\{a_i\}\{b_i\}) = \varphi(\{a_i b_i\}) = \sum a_i b_i = \sum a_i \sum b_i = \varphi(\{a_i\})\varphi(\{b_i\})$, where the third equality follows from $a_i b_j \in J_i \cap J_j = \{0\}$ whenever $i \neq j$.
(iii) φ is injective: Suppose $\varphi(\{a_i\}) = \sum a_i = 0$. Then, for each i, we have $a_i = -\sum_{j \neq i} a_j \in J_i \cap \sum_{j \neq i} J_j = \{0\}$ and therefore $a_i = 0$.
(iv) φ is surjective: This is obvious from $R = J_1 + \cdots + J_n$.
From (i)-(iv), we conclude φ is an isomorphism of rings and $R \cong \prod_{i=1}^{n} J_i$. (cf. Theorem 9 [p171, Dummit & Foote])

Question 2 Look up Zorn's lemma on partially ordered sets (*e.g.* Dummit and Foote Appendix I). Let R be a ring with 1 and at least one proper ideal (an ideal other than R or $\{0\}$). Now prove that R has a maximal proper ideal (*i.e.* not contained in any other proper ideal)[1].

Solution The proof goes exactly the same as that of Proposition 11 [p254, Dummit & Foote].

Question 3 *Frobenius homomorphism* Let R be a commutative ring with identity and prime characteristic p. Show that the map

$$\varphi : R \longrightarrow R, \, , \quad r \longmapsto r^p \, ,$$

[1]Hint: study "chains" in $\{A | I \subseteq A \subseteq R\}$ for some proper ideal I.

is a ring homomorphism.

Solution See Proposition 35 [p548, Dummit & Foote], whose proof actually works for a commutative ring with identity of characteristic p. The field axiom is used only when to show injectivity.

Question 4 Find all ideals in $\mathbb{Z}$.

Solution See Examples (2) [p243, Dummit & Foote] and Examples (2) [p252, Dummit & Foote]. Recall also Theorem 7 [p58, Dummit & Foote].