250A Homework 1

Solution by Jaejeong Lee

Question 1 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator. Show that T has 1 and 2 dimensional invariant subspaces.

Solution The characteristic polynomial of T is real cubic, so it has at least one real root λ and an eigenvector $v \in \mathbb{R}^3$ such that $Tv = \lambda v$. Note that $span\{v\} \subset ker(T - \lambda I)$ is a 1-dimensional invariant subspace of $\mathbb{R}^3$ and $rank(T - \lambda I) = \dim\, im(T - \lambda I) = 3 - \dim\, ker(T - \lambda I) < 3$. Now if $rank(T - \lambda I) = 0$, then $T = \lambda I$ and any 2-dimensional subspace of $\mathbb{R}^3$ is invariant under T. If $rank(T - \lambda I) = 1$, then $ker(T - \lambda I)$ is a 2-dimensional invariant subspace, since for $v \in ker(T - \lambda I)$ we have $(T - \lambda I)(Tv) = T(Tv - \lambda v) = T(0) = 0$ and hence $T(ker(T - \lambda I)) \subset ker(T - \lambda I)$. If $rank(T - \lambda I) = 2$, then $im(T - \lambda I)$ is a 2-dimensional invariant subspace since $T((T - \lambda I)v) = (T - \lambda I)(Tv) \in im(T - \lambda I)$. Therefore, in any case, T has a 2-dimensional invariant subspace, too.

Question 2 Let M be the space of 3 matrices. What is $\dim(M)$? Now define the linear operator $T : M \to M$ by

$$M \ni m \overset{T}{\longmapsto} \frac{1}{2}\left[\begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix} m + m \begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix}\right].$$

Compute $\det T$.

Solution The standard basis for M is $\{e_{ij} \mid 1 \leq i, j \leq 3\}$, where e_{ij} is a matrix unit with its only nonzero entry being 1 at (i, j). Thus $\dim(M) = 9$. We compute

$$T\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2m_{11} & 3m_{12} & 2m_{13} \\ 3m_{21} & 4m_{22} & 3m_{23} \\ 2m_{31} & 3m_{32} & 2m_{33} \end{pmatrix},$$

so the eigenvectors of T are $\{e_{ij} \mid 1 \leq i, j \leq 3\}$ and the corresponding eigenvalues are $\{1, \frac{3}{2}, 1, \frac{3}{2}, 2, \frac{3}{2}, 1, \frac{3}{2}, 1\}$. Therefore, $\det T = \prod(eigenvalues) = \frac{81}{8}$.

Question 3 *Van der Monde Determinant.* Let A be the $(n \times n)$ matrix with entries

$$A_{ij} = (a_i)^{j-1}\,.$$

Show that $\det A = \prod_{i<j}(a_j - a_i)$.

Solution Note that

$$A = \begin{pmatrix} 1 & a_1 & (a_1)^2 & \cdots & (a_1)^{n-1} \\ 1 & a_2 & (a_2)^2 & \cdots & (a_2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & (a_n)^2 & \cdots & (a_n)^{n-1} \end{pmatrix}.$$

We use induction on n. When $n = 2$, we verify

$$\det \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} = a_2 - a_1.$$

Assume now the assertion is true up to $n-1$ and let

$$f(x) = \det \begin{pmatrix} 1 & a_1 & (a_1)^2 & \cdots & (a_1)^{n-1} \\ 1 & a_2 & (a_2)^2 & \cdots & (a_2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & (a_{n-1})^2 & \cdots & (a_{n-1})^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{pmatrix}.$$

Since $f(x)$ is a polynomial of degree $n-1$ and $f(a_i) = 0$ for $1 \leq i \leq n-1$, we have

$$\begin{aligned} &f(x) \\ &= (-1)^{n+n} \det \begin{pmatrix} 1 & a_1 & \cdots & (a_1)^{n-2} \\ 1 & a_2 & \cdots & (a_2)^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & \cdots & (a_{n-1})^{n-2} \end{pmatrix} (x-a_1)(x-a_2)\cdots(x-a_{n-1}) \\ &= \left(\prod_{1 \leq i < j \leq n-1} (a_j - a_i) \right) (x-a_1)(x-a_2)\cdots(x-a_{n-1}), \end{aligned}$$

by the induction hypothesis. Therefore, we finally get

$$
\begin{aligned}
&\det\begin{pmatrix} 1 & a_1 & (a_1)^2 & \cdots & (a_1)^{n-1} \\ 1 & a_2 & (a_2)^2 & \cdots & (a_2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & (a_n)^2 & \cdots & (a_n)^{n-1} \end{pmatrix} \\
&= f(a_n) \\
&= \left(\prod_{1\le i<j\le n-1}(a_j-a_i)\right)(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1}) \\
&= \prod_{1\le i<j\le n}(a_j-a_i)
\end{aligned}
$$

and the assertion is true for n.

Question 4 *(Anti)commutators.* Let V be a finite dimensional vector space. Show that the mapping

$$[\cdot,\cdot] : L(V)\times L(V)\to L(V)$$

where

$$[\cdot,\cdot] : (M,N)\mapsto MN-NM\equiv[M,N]\,,$$

obeys the Leibnitz rule $[M,NR]=[M,N]R+N[M,R]$. In addition, verify the Jacobi identity

$$[M,[N,R]]+[N,[R,M]]+[R,[M,N]]=0\,.$$

Generalize the above laws to the mapping $\{\cdot,\cdot\} : (M,N)\mapsto MN+NM\equiv\{M,N\}$. Include also new rules which mix both operations.

Solution We want to find a kind of the Leibnitz rule

$$F(N*R)=F(N)*R+N*F(R)$$

in cases $F(\,\cdot\,)=[M,\,\cdot\,]$ or $\{M,\,\cdot\,\}$ and $N*R=NR$, $[N,R]$, or $\{N,R\}$. For example, when $F(\,\cdot\,)=[M,\,\cdot\,]$ and $N*R=NR$, we have

$$[M,NR]=[M,N]R+N[M,R] \tag{1}$$

and when $F(\,\cdot\,) = [M, \,\cdot\,]$ and $N * R = [N, R]$, we have

$$[M, [N, R]] = [[M, N], R] + [N, [M, R]]. \tag{2}$$

Now I present a *trick* to find such rules. (It may not be a trick but a principle. Because I don't know why it works, to me, it is a trick.)

> Matrices can be assigned two attributes, even and odd. If M is assigned odd, we mark it M', and if even, leave it as it is.
> **Step1** Write down (1) or (2).
> *e.g.* $[M, NR] = [M, N]R + N[M, R]$
> **Step2** Assign attributes to M, N, R arbitrarily.
> *e.g.* $[M', N'R] = [M', N']R + N'[M', R]$
> **Step3** Regard (odd,odd)-pair and (even,even)-pair as even and other pairs as odd.
> *e.g.* $N'R$ is odd, so $[M', N'R]$ is even. $[M', N']$ is even and $[M', R]$ is odd.
> **Step4** If you fi nd [odd, odd], replace $[\,\cdot\,, \,\cdot\,]$ by $\{\,\cdot\,, \,\cdot\,\}$.
> *e.g.* $\{M', N'R\} = \{M', N'\}R + N'[M', R]$
> **Step5** If an odd, as an operator, goes past another odd, then replace $+$ by $-$.
> *e.g.* $\{M', N'R\} = \{M', N'\}R - N'[M', R]$ (M' goes past N')
> **Step6** Remove markings $'$ and declare you found a rule.
> *e.g.* $\{M, NR\} = \{M, N\}R - N[M, R]$

Example1. $[M, NR] = \{M, N\}R - N\{M, R\}$.

> **Step1** $[M, NR] = [M, N]R + N[M, R]$.
> **Step2** $[M', N'R'] = [M', N']R' + N'[M', R']$
> **Step3** Observe $[M', N']$ and $[M', R']$.
> **Step4** $[M', N'R'] = \{M', N'\}R' + N'\{M', R'\}$
> **Step5** $[M', N'R'] = \{M', N'\}R' - N'\{M', R'\}$
> **Step6** $[M, NR] = \{M, N\}R - N\{M, R\}$

Example2. $[M, \{N, R\}] = [\{M, N\}, R] - [N, \{M, R\}]$.

> **Step1** $[M, [N, R]] = [[M, N], R] + [N, [M, R]]$.
> **Step2** $[M', [N', R']] = [[M', N'], R'] + [N', [M', R']]$
> **Step3** Observe $[N', R']$, $[M', N']$, and $[M', R']$.
> **Step4** $[M', \{N', R'\}] = [\{M', N'\}, R'] + [N', \{M', R'\}]$
> **Step5** $[M', \{N', R'\}] = [\{M', N'\}, R'] - [N', \{M', R'\}]$
> **Step6** $[M, \{N, R\}] = [\{M, N\}, R] - [N, \{M, R\}]$

Example3. $[M, \{N, R\}] = \{[M, N], R\} + \{N, [M, R]\}$.

Step1 $[M, [N, R]] = [[M, N], R] + [N, [M, R]]$.
Step2 $[M, [N', R']] = [[M, N'], R'] + [N', [M, R']]$
Step3 $[N', R']$ is even, $[M, N']$ and $[M, R']$ are odd.
Step4 $[M, \{N', R'\}] = \{[M, N'], R'\} + \{N', [M, R']\}$
Step5 $[M, \{N', R'\}] = \{[M, N'], R'\} + \{N', [M, R']\}$ (No change)
Step6 $[M, \{N, R\}] = \{[M, N], R\} + \{N, [M, R]\}$

Find more on your own. Note that you never get $\{M, \{N, R\}\}$ because of **Step3** and **Step4**.

Question 5 *Baker Campbell Hausdorff Formula.* Let V be a finite dimensional vector space and $M, N \in L(V)$. Show that

$$\exp(M)\exp(N) = \exp(M + N + \frac{1}{2}[M, N]) \, ,$$

if $0 = [M, [M, N]] = [N, [M, N]]$. *Hint: Develop and solve a differential equation for* $R(\lambda) \equiv \exp(\lambda M)\exp(\lambda N) \in L(V)$.

Solution We need the identity $e^{\lambda M} N e^{-\lambda M} = e^{\lambda[M, \cdot\,]} N$ (proved below) to see

$$\begin{aligned} \frac{d}{d\lambda} R(\lambda) &= M R(\lambda) + e^{\lambda M} N e^{-\lambda M} R(\lambda) \\ &= \left(M + e^{\lambda[M, \cdot\,]} N\right) R(\lambda) \\ &= (M + N + \lambda[M, N])\, R(\lambda), \end{aligned}$$

the last equality coming from the commutativity assumptions. Since $R(0) = 0$ we get the unique solution

$$R(\lambda) = e^{\lambda(M+N) + \frac{1}{2}\lambda^2 [M,N]}.$$

(Verify this. We need the commutativity assumptions again. To solve $f'(x) = (a + bx) f(x)$ we observe $(\log f(x))' = a + bx$.) Plugging $\lambda = 1$ we get the desired equality. To show $e^{\lambda M} N e^{-\lambda M} = e^{\lambda[M, \cdot\,]} N$, let $L_M(\,\cdot\,) = [M, \,\cdot\,]$ and compute the Taylor expansion for $e^{\lambda M} N e^{-\lambda M}$;

$$\begin{aligned}
\frac{d}{d\lambda} e^{\lambda M} N e^{-\lambda M} &= M e^{\lambda M} N e^{-\lambda M} - e^{\lambda M} N e^{-\lambda M} M \\
&= [M,\, e^{\lambda M} N e^{-\lambda M}] \\
&= L_M (e^{\lambda M} N e^{-\lambda M}), \\
\frac{d^2}{d\lambda^2} e^{\lambda M} N e^{-\lambda M} &= [M,\, \frac{d}{d\lambda} e^{\lambda M} N e^{-\lambda M}] \\
&= L_M L_M (e^{\lambda M} N e^{-\lambda M}),
\end{aligned}$$

and so on. Therefore

$$\begin{aligned}
e^{\lambda M} N e^{-\lambda M} &= N + \lambda L_M(N) + \frac{1}{2}\lambda^2 L_M L_M(N) + \cdots \\
&= e^{\lambda L_M} N.
\end{aligned}$$