250A Homework 2

Solution by Jaejeong Lee

Question 1 Let G be a group and suppose $x^2 = 1\, \forall x \in G$. Prove that G is abelian.

Solution Let $a, b \in G$. By assumption, we have

$$aba^{-1}b^{-1} = aba^{-1}b^{-1}(ba)^2 = ab^2a = a^2 = 1,$$

that is, $ab = ba$. Thus, G is abelian.

Question 2 Let $G = \{x_1, \ldots, x_n\}$ be a finite abelian group. Prove

$$(x_1 \cdots x_n)^2 = 1.$$

Solution Note that the map $I : G \to G$ defined by $I(g) = g^{-1}$ is a bijection. (It is an isomorphism in case G is abelian.) Since G is abelian, we have

$$\begin{aligned}
(x_1 \cdots x_n)^2 &= (x_1 \cdots x_n)(I(x_1) \cdots I(x_n)) \\
&= (x_1 I(x_1)) \cdots (x_n I(x_n)) \\
&= (x_1 {x_1}^{-1}) \cdots (x_n {x_n}^{-1}) \\
&= 1.
\end{aligned}$$

Question 3 Find all groups of order 7 or less.

Solution Groups of prime order are cyclic by Corollary 10 [p90, Dummit & Foote]. Suppose $|G| = 4$. If G has an element of order 4, then G is isomorphic to Z_4. If there is no element of order 4 in G, then every nontrivial elements of G have order 2 and G is isomorphic to the Klein 4-group V_4. Now suppose $|G| = 6$. As before, G is isomorphic to Z_6 or G has an element r of order 3. (If every nontrivial elements of G have order 2, consider $\langle a \rangle$ for some $1 \neq a \in G$. By Question 1, G is abelian and thus $\langle a \rangle$ is normal in G. Now the quotient group $G/\langle a \rangle$ is of order 3 and hence cyclic, but it has no order 3 element; a contradiction!) Consider $\langle r \rangle$, which is normal being of index 2, and let $m \notin \langle r \rangle$. Since $G/\langle r \rangle = \{\langle r \rangle, m\langle r \rangle\} = \{\langle r \rangle, \langle r \rangle m\}$, m cannot be of order 3, that is, $|m| = 2$, and we have (1) $mr = rm$ or (2) $mr = r^2m$: (1) G is abelian and is isomorphic to $Z_2 \times Z_3 \simeq Z_6$, (2) $G = \langle r, m \mid r^3 = m^2 = 1, mrm^{-1} = r^{-1} \rangle \simeq S_3 \simeq D_3$. In summary, groups of order 7 or less are : $\{1\}, Z_2, Z_3, Z_4, V_4, Z_5, Z_6, S_3, Z_7$.

Question 4 Let $\mathbb{R}^+$ and $\mathbb{R}^*$ be the group of real numbers under addition and non-zero real numbers under multiplication, respectively. Show that these groups are not isomorphic.

Solution Every nontrivial element of $\mathbb{R}^+$ is of infinite order. On the other hand, $\mathbb{R}^*$ has a nontrivial element of order 2, namely, -1. Therefore, these two groups cannot be isomorphic.

Question 5 Show that the set of nth roots of unity $G = \{z \in \mathbb{C} | z^n = 1, n \in \mathbb{N}\}$ is a group under multiplication but not addition.

Solution Let $z, w \in G$. Then $z^m = w^n = 1$ for some $m, n \in \mathbb{N}$. Since

$$(zw^{-1})^{mn} = (z^m)^n (w^n)^{-m} = 1,$$

we see $zw^{-1} \in G$. By Proposition 1 [p47, Dummit & Foote], $(G, \cdot)$ is a subgroup of the group $\mathbb{C}^*$ of non-zero complex numbers. In particular, $(G, \cdot)$ is a group in itself. On the other hand, $(G, +)$ is not a group because it is not closed under addition; $1 + 1 \notin G$.

Question 6 Let $x \in G$ be a group. Show that $|x| = n < \infty$ implies that $1, x, x^2, \ldots, x^{n-1}$ are distinct.

Solution Suppose on the contrary that $x^i = x^j$ for some $0 \leq i < j \leq n-1$. Then $x^{j-i} = 1$; a contradiction since $0 < j - i < n$.

Question 7 Suppose $\mathcal{N}$ is a nilpotent operator on a vector space V. In this case V is a direct sum of cyclic subspaces. Show that the number of summands is $\dim \ker \mathcal{N}$.

Solution Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ be a decomposition of V into a direct sum of cyclic subspaces of $\mathcal{N}$. If we denote $V_i = \langle e_i, \mathcal{N}e_i, \mathcal{N}^2 e_i, \cdots, \mathcal{N}^{m_i - 1} e_i \rangle$ for

$1 \leq i \leq k$, then $\ker(\mathcal{N}|_{V_i}) = \langle \mathcal{N}^{m_i - 1} e_i \rangle$, so $\dim \ker(\mathcal{N}|_{V_i}) = 1$. Since

$$\begin{aligned} \ker \mathcal{N} &= \ker \mathcal{N} \cap V \\ &= \ker \mathcal{N} \cap (V_1 \oplus \cdots \oplus V_k) \\ &= (\ker \mathcal{N} \cap V_1) \oplus \cdots \oplus (\ker \mathcal{N} \cap V_k) \\ &= \ker(\mathcal{N}|_{V_1}) \oplus \cdots \oplus \ker(\mathcal{N}|_{V_k}), \end{aligned}$$

we have $\dim \ker \mathcal{N} = k$.