250A Homework 3

Solution by Jaejeong Lee

Question 1 Definition: $\operatorname{tor}(G) = \{g \in G | g^n = 1 \text{for some } n \in \mathbb{N}\}$. In addition a group G is called torsion-free if $\operatorname{tor}(G) = \{1\}$. Let Γ be a group which contains a torsion-free subgroup Γ_0 of index $n < \infty$. Show that Γ does not contain any element whose order is finite and strictly larger than n.

Solution Let $\gamma \in \Gamma$ be of finite order. Since $[\Gamma : \Gamma_0] = n$, there exist $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$ and $\gamma^i \Gamma_0 = \gamma^j \Gamma_0$ (Pigeonhole principle), and thus also $k \in \{1, \ldots, n\}$ such that $\gamma^k \in \Gamma_0$. As γ has finite order and Γ_0 is torsion-free, we have $\gamma^k = 1$, and hence $|\gamma| \leq n$.

Question 2 Show that $\operatorname{tor}(G) \leq G$ if G is abelian.

Solution Let $g, h \in \operatorname{tor}(G)$ then $g^m = h^n = 1$ for some $m, n \in \mathbb{N}$. Since G is abelian, we have

$$(gh^{-1})^{mn} = (g^m)^n (h^n)^{-m} = 1,$$

so $gh^{-1} \in \operatorname{tor}(G)$. Since $1 \in \operatorname{tor}(G) \neq \emptyset$, $\operatorname{tor}(G)$ is a subgroup of G by Proposition 1 [p47, Dummit & Foote].

Question 3 Show that the order of a permutation is the least common multiple of the lengths of its disjoint cycles.

Solution Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ be the cycle decomposition of σ into disjoint cycles and let r_i be the length of σ_i. Then $|\sigma_i| = r_i$ for $1 \leq i \leq k$. Suppose now $\sigma^r = 1$ for some r. Since disjoint cycles commute, we have $1 = \sigma^r = {\sigma_1}^r {\sigma_2}^r \cdots {\sigma_k}^r$. Observe that ${\sigma_i}^r$'s are again disjoint cycles, so we see ${\sigma_i}^r = 1$ and hence $r_i | r$ for all i. From the minimality property, we get $|\sigma| = \text{l.c.m.}\{r_1, r_2, \cdots, r_k\}$.

Question 4 Find all normal subgroups of S_4.

Solution [cf. pp125-128, Dummit & Foote] Since cycle types do not change under conjugation (Proposition 11), each conjugacy class of S_4 is made up of elements of the same cycle type. Observe that representatives of the cycle types of elements in S_4 can be taken to be

$$1, (12), (123), (12)(34), (1234),$$

each representing 1,6,8,3,6 elements in its conjugacy class, respectively. Since normal subgroups are the union of conjugacy classes (p127), the possible order of normal subgroups are

$$1 = 1, \quad 4 = 1 + 3, \quad 12 = 1 + 8 + 3, \quad 24 = 1 + 6 + 8 + 3 + 6.$$

Therefore, normal subgroups of S_4 are

$$1, \quad \{(1), (12)(34), (13)(24), (14)(23)\}, \quad A_4, \quad S_4.$$

Question 5 Show that $K \leq H \leq G$ and $K \trianglelefteq G \Rightarrow K \trianglelefteq H$. Now, a permutation is called even if it can be written as an even number of transpositions (e.g. (132)=(12)(13) is even). The alternating group A_n is the subgroup of the symmetric group S_n consisting of only even permutations. Study the group A_4 to decide whether $K \trianglelefteq H \trianglelefteq G \Rightarrow K \trianglelefteq G$.

Solution The first assertion is clear by definition. Let

$$K = \{(1), (12)(34)\}, \quad H = \{(1), (12)(34), (13)(24), (14)(23)\}, \quad G = A_4.$$

From Question 4 and the first assertion, we have $H \trianglelefteq G$. Being of index two, $K \trianglelefteq H$. But $(123)[(12)(34)](123)^{-1} = (23)(14) \notin K$, so $K \ntrianglelefteq G$.

Question 6 Show that $Sl(n, \mathbb{R}) \trianglelefteq Gl(n, \mathbb{R})$.

Solution Since $\det(AB) = \det(A)\det(B)$, the determinant function

$$\det : Gl(n, \mathbb{R}) \to (\mathbb{R} \setminus \{0\}, \cdot\,)$$

is a group homomorphism. Thus, $Sl(n, \mathbb{R}) = \ker(\det)$ is a normal subgroup of $Gl(n, \mathbb{R})$.