250A Homework 5

Solutions by Jaejeong Lee

Question 1 Subgroups $H \leq S_4$ act on $S \equiv \{1,2,3,4\}$ in the natural way. For each subgroup H below, compute the orbits and stabilizer of each $s \in S$:

(i) $H = \langle (123) \rangle$

(ii) $H = \langle (1234) \rangle$

(iii) $H = \langle (12), (34) \rangle$

(iv) $H = D_8$

(v) $H = A_4$

Solution (i) $S = \{1,2,3\} \cup \{4\}$, $H_1 = H_2 = H_3 = \{(1)\}$, $H_4 = H$. (ii) $S = S$, $H_1 = H_2 = H_3 = H_4 = \{(1)\}$. (iii) $S = \{1,2\} \cup \{3,4\}$, $H_1 = H_2 = \langle (34) \rangle$, $H_3 = H_4 = \langle (12) \rangle$. (iv) Let $H = \langle (1234), (24) \rangle$. $S = S$, $H_1 = H_3 = \langle (24) \rangle$, $H_2 = H_4 = \langle (13) \rangle$. (v) $S = S$, $H_1 = \langle (234) \rangle$, $H_2 = \langle (134) \rangle$, $H_3 = \langle (124) \rangle$, $H_4 = \langle (123) \rangle$.

Question 2 Let $G = \mathbb{R} \ni r$ be the group of real numbers under addition. Show that

(i) $S = \mathbb{R}^n \ni x \mapsto x + rv$ ($v \in \mathbb{R}^n$ fixed),

(ii) $S = \mathbb{C} \ni \rho e^{i\theta} \mapsto \rho e^{i(\theta+r)}$

are group actions and describe their orbits geometrically. What do stabilizers look like?

Solution (i) Orbits are lines parallel to $v(\neq 0)$ and they foliate $\mathbb{R}^n$. Stabilizers are all trivial. (ii) Orbits are the origin and concentric circles centered at the origin. Stabilizers of nonzero complex numbers are all $\mathbb{Z} \simeq \langle 2\pi \rangle < \mathbb{R}$.

Question 3 Let $H \leq G$ with $[G : H] = n$. Show that G has a proper normal subgroup of index at most $n!$.

Solution Consider the action of G on the set A of left cosets of H by left multiplication. Let $\pi_H : G \to S_A$ be the corresponding permutation representation. The kernel $\ker \pi_H$ is a normal subgroup of G and is of index $[G : \ker \pi_H] = |G/\ker \pi_H| = |\mathrm{im}\, \pi_H| \leq |S_A|$. But $|S_A| = n!$ since $[G : H] = |A| = n$. Note also $\ker \pi_H \subset H$. Therefore $[G : \ker \pi_H] \geq n$ and $\ker \pi_H$ is proper. (cf. Theorem 3 [p119, Dummit & Foote])

Question 4 Find the center of $Gl(2, \mathbb{R})$, the dihedral groups, and $SO(3, \mathbb{R})$.

Solution (i) Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(Gl(2, \mathbb{R}))$. Let $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. From $AE = EA$ and $AF = FA$, we get $a = d$ and $b = c = 0$. It is now clear that $Z(Gl(2, \mathbb{R})) = \{aI \mid a \in \mathbb{R} \setminus \{0\}\}$. (ii) Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs^{-1} = r^{-1} \rangle = \{1, r, r^2, \ldots, r^{n-1}, s, rs, r^2 s, \ldots, r^{n-1}s\}$. Observe that

$$(r^i s)(r^k)(r^i s)^{-1} = r^{-k} \neq r^k \text{ for } k \neq n/2$$

$$(r^i)(r^k s)(r^i)^{-1} = r^{k+2i} s \neq r^k s \text{ for } i \neq 0.$$

Therefore, $Z(D_{2n}) = \{1, r^{n/2}\}$ for n even and $Z(D_{2n}) = \{1\}$ for n odd. (iii) Suppose $A = (a_{ij}) \in Z(SO(3, \mathbb{R}))$. Let

$$E = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \; F = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \text{ and } G = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}.$$

From $AE = EA$ and $AF = FA$, we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$. From $AG = GA$, we get $a_{11} = a_{22} = a_{33}$. Since $\det A = 1$, we see $A = I$ and $Z(SO(3, \mathbb{R})) = \{I\}$.

Question 5 Show that $G/Z(G)$ cyclic $\Rightarrow$ G is abelian. Show that if p is prime, then all groups of order p^2 are abelian.

Solution (i) Let $G/Z(G) = \langle gZ(G) \rangle$ for some $g \in G$. If $a, b \in G$, then $a \in g^k Z(G)$ and $b \in g^l Z(G)$ for some k, l and thus, $a = g^k x$ and $b = g^l y$ for some $x, y \in Z(G)$. We now have

$$ab = (g^k x)(g^l y) = g^{k+l} xy = (g^l y)(g^k x) = ba,$$

since $x, y \in Z(G)$. Therefore G is abelian. (ii) (cf. [p125, Dummit & Foote]) Let G be a group of order p^2. By Theorem 8, $Z(G) \neq \{1\}$. Since $|G/Z(G)| = p$ or 1, $G/Z(G)$ is cyclic and by (i) G is abelian. (cf. Corollary 9).

Question 6 *Research problem – answers must be less than 1 page*. What are (i) a Lie group, (ii) a Lie algebra and (iii) a Dynkin diagram?