250A Homework 6

Solutions by Jaejeong Lee

Question 1 Let G act on the set S. Show that if $s, t \in S$ belong to the same orbit, their stabilizers are conjugate subgroups of G.

Solution Suppose $g_0 s = t$ for some $g_0 \in G$. Then, we have

$$\begin{aligned} G_t &= \{g \in G \mid gt = t\} \\ &= \{g \in G \mid g(g_0 s) = g_0 s\} \\ &= \{g \in G \mid ({g_0}^{-1} g g_0)s = s\} \\ &= \{g \in G \mid {g_0}^{-1} g g_0 \in G_s\} \\ &= \{g \in G \mid g \in g_0 G_s {g_0}^{-1}\} \\ &= g_0 G_s {g_0}^{-1}. \end{aligned}$$

Question 2 Let x and y be conjugate elements in a finite group G. Show that the number of distinct $g \in G$ with $g^{-1}xg = y$ equals the order of the normalizer of x.

Solution Since x and y are conjugate, ${g_0}^{-1} x g_0 = y$ for some $g_0 \in G$. We have

$$\begin{aligned} \{g \in G \mid g^{-1}xg = y\} &= \{g \in G \mid g^{-1}xg = {g_0}^{-1} x g_0\} \\ &= \{g \in G \mid x(g{g_0}^{-1}) = (g{g_0}^{-1})x\} \\ &= \{g \in G \mid g{g_0}^{-1} \in C_G(x)\} \\ &= \{g \in G \mid g \in C_G(x) g_0\} \\ &= C_G(x) g_0 \end{aligned}$$

and the claim follows from $|C_G(x)g_0| = |C_G(x)|$ and $C_G(x) = N_G(\{x\})$.

Question 3 Which finite groups have precisely one or two conjugacy classes?

Solution (i) Suppose G has only one conjugacy class. Then every element of G is conjugate to the identity. Thus $G = \{1\}$. (ii) Suppose $|G| = n$ and G has exactly two conjugacy classes. One of the two conjugacy classes must be $\{1\}$, so the other is $G \setminus \{1\}$. Since the order of a conjugacy class divides $|G|$, we have $n - 1 | n$. Thus $n = 2$ and $G \simeq \mathbb{Z}_2$.

Question 4 Fix $x \in G$ a group. Show that $G \ni g \mapsto xgx^{-1} \in G$ is an automorphism of G and in turn demonstrate that $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$.

Solution This is a special case of Proposition 13 [p133, Dummit & Foote] with $H = G$.